

# Contents

<b>Foreword by Jan Mycielski</b>	<i>page xi</i>
<b>Preface</b>	xiii
<b>Preface to the Paperback Edition</b>	xvii
<b>Part I Paradoxical Decompositions, or the Nonexistence of Finitely Additive Measures</b>	<b>1</b>
<b>Chapter 1 Introduction</b>	<b>3</b>
Historical background of paradoxical decompositions and a formal definition. The central example: free non-Abelian groups. Connections with non-Lebesgue measurable sets. A paradoxical subset of the plane. Paradoxical actions of free groups. The possibility of free groups being the only paradoxical ones.	
<b>Chapter 2 The Hausdorff Paradox</b>	<b>15</b>
A free non-Abelian group of rotations of a sphere. The nonexistence of certain finitely additive measures in $\mathbf{R}^n$ , $n \geq 3$ .	

<b>Chapter 3</b>	<b>The Banach-Tarski Paradox: Duplicating Spheres and Balls</b>	<b>21</b>
	<p>Congruence by dissection and the Bolyai-Gerwien Theorem on polygons of equal area.</p> <p>Equidecomposability: a set-theoretic form of dissection. A generalization of the Schröder-Bernstein Theorem, and an application to dissections of polygons.</p> <p>Absorbing a countable subset of a sphere by a rotation.</p> <p>The Banach-Tarski Paradox: duplicating spheres and balls with rotations and translations. Strong form of the paradox: the equidecomposability of arbitrary bounded sets with nonempty interiors in <math>\mathbf{R}^3</math>. Some open questions derived from the Banach-Tarski Paradox.</p>	
<b>Chapter 4</b>	<b>Locally Commutative Actions: Minimizing the Number of Pieces in a Paradoxical Decomposition</b>	<b>34</b>
	<p>Free non-Abelian groups are paradoxical using four pieces. Locally commutative actions of free groups. A sphere is paradoxical using four and not fewer pieces, a ball using five and not fewer pieces. Four-piece paradoxical decompositions imply the existence of free subgroups. Arbitrary systems of congruences. Divisibility of a sphere into rotationally congruent pieces. Allowing the antipodal map allows more systems of congruences to be solved.</p>	
<b>Chapter 5</b>	<b>Higher Dimensions and Non-Euclidean Spaces</b>	<b>52</b>
	<p>Free subgroups of higher dimensional rotation groups, and the type of paradoxes they cause. Free groups of isometries of <math>\mathbf{R}^n</math>, <math>n \geq 3</math>, acting without fixed points. Paradoxes in non-Euclidean spaces; a constructive paradoxical decomposition of the hyperbolic plane. A result on tetrahedral snakes.</p>	
<b>Chapter 6</b>	<b>Free Groups of Large Rank: Getting a Continuum of Spheres from One</b>	<b>73</b>
	<p>Solving large systems of congruences using free groups of large rank. Using a continuum of algebraically independent numbers to construct large free groups of spherical, Euclidean and non-Euclidean isometries. An alternate approach using analytic functions. The transfinite duplication of a sphere. The anomaly of the hyperbolic plane. The case of hyperbolic 3-space. Free</p>	

semigroups of large rank, and the planar decompositions they cause. Sets in metric and Euclidean spaces congruent to proper subsets. The analogous problem in groups.

<b>Chapter 7</b>	<b>Paradoxes in Low Dimensions</b>	<b>96</b>
	A free non-Abelian group of linear transformations of determinant 1 in $\mathbf{R}^2$ . A paradox in the plane using affine, area-preserving transformations. Tarski's Circle-Squaring Problem. Restricting the problem to pieces made by cutting on Jordan curves. Allowing similarity transformations that magnify only a little. A paradox on the line using linear fractional transformations.	
<b>Chapter 8</b>	<b>The Semigroup of Equidecomposability Types</b>	<b>109</b>
	How to add equidecomposability tapes. An application to locally commutative actions. A cancellation law in the type semigroup. An application to spheres. The role of the Axiom of Choice in the cancellation law. Equidecomposability with some natural restrictions on the pieces.	
<b>Part II</b>	<b>Finitely Additive Measures, or the Nonexistence of Paradoxical Decompositions</b>	<b>123</b>
<b>Chapter 9</b>	<b>Transition</b>	<b>125</b>
	Tarski's Theorem equating the nonexistence of paradoxes with the existence of finitely additive invariant measures. A choiceless version of Tychonoff's Theorem. Measures on the algebra of sets with the Property of Baire; equivalent formulations of Marczewski's Problem. Uniqueness of Jordan and Lebesgue measure. Equidecomposability using countably many pieces, with and without restrictions on the pieces.	
<b>Chapter 10</b>	<b>Measures in Groups</b>	<b>146</b>
	Amenable groups and left-invariant means. All elementary groups are amenable. A characterization of amenable matrix groups. Constructing measures invariant with respect to an amenable group. The absence of paradoxes in $\mathbf{R}^1$ and $\mathbf{R}^2$ . Properties of groups that are equivalent to amenability.	

<b>Chapter 11</b>	<b>Applications of Amenability: Marczewski Measures and Exotic Measures</b>	<b>165</b>
	The existence of invariant measures that vanish on meager sets. Invariant measures that treat similarities properly. The existence of exotic measures in $\mathbf{R}^1$ and $\mathbf{R}^2$ . The use of Property $T$ to disprove the existence of exotic measures in higher dimensions. Exotic measures and paradoxical decompositions modulo an ideal. Paradoxes in $\mathbf{R}^n$ using measurable sets. Characterizing the Euclidean isometry groups with respect to which invariant measures exist.	
<b>Chapter 12</b>	<b>Growth Conditions in Groups and Supramenability</b>	<b>188</b>
	Supramenable groups and free subsemigroups. Growth conditions in groups: slow growth implies supramenability. The nonexistence of a paradoxical subset of the real line. Polynomial growth and the Milnor-Wolf Conjecture. A characterization of the groups of isometries with respect to which a paradoxical subset of $\mathbf{R}^n$ exists. Two-piece paradoxical decompositions and free subsemigroups.	
<b>Chapter 13</b>	<b>The Role of the Axiom of Choice</b>	<b>207</b>
	Consistency results in measure theory: The Banach-Tarski Paradox is not a theorem of ZF. Neither are some basic results about amenable groups. Eliminating the Axiom of Choice from geometrical results. Foundational implications of the Banach-Tarski Paradox.	
<b>Appendix A</b>	<b>Euclidean Transformation Groups</b>	<b>222</b>
<b>Appendix B</b>	<b>Jordan Measure</b>	<b>227</b>
<b>Appendix C</b>	<b>Unsolved Problems</b>	<b>229</b>
<b>Addendum to Second Printing</b>		<b>234</b>
<b>References</b>		<b>246</b>
<b>List of Symbols</b>		<b>248</b>
<b>Index</b>		<b>250</b>