

Contents

Preface to Fifth Edition	xv
Acknowledgements	xvii
1 Preliminaries: Computer Strategies	1
1.1 Introduction	1
1.2 Hardware	2
1.3 Memory Management	2
1.4 Vector Processors	3
1.5 Multi-core Processors	3
1.6 Co-processors	4
1.7 Parallel Processors	4
1.8 Applications Software	5
1.8.1 Compilers	5
1.8.2 Arithmetic	6
1.8.3 Conditions	7
1.8.4 Loops	8
1.9 Array Features	9
1.9.1 Dynamic Arrays	9
1.9.2 Broadcasting	9
1.9.3 Constructors	9
1.9.4 Vector Subscripts	10
1.9.5 Array Sections	11
1.9.6 Whole-array Manipulations	11
1.9.7 Intrinsic Procedures for Arrays	11
1.9.8 Modules	12
1.9.9 Subprogram Libraries	13
1.9.10 Structured Programming	15
1.10 Third-party Libraries	17
1.10.1 BLAS Libraries	17
1.10.2 Maths Libraries	17
1.10.3 User Subroutines	18
1.10.4 MPI Libraries	18
1.11 Visualisation	18
1.11.1 Starting ParaView	19
1.11.2 Display Restrained Nodes	20

1.11.3	<i>Display Applied Loads</i>	21
1.11.4	<i>Display Deformed Mesh</i>	21
1.12	Conclusions	23
	References	24
2	Spatial Discretisation by Finite Elements	25
2.1	Introduction	25
2.2	Rod Element	25
2.2.1	<i>Rod Stiffness Matrix</i>	25
2.2.2	<i>Rod Mass Element</i>	28
2.3	The Eigenvalue Equation	28
2.4	Beam Element	29
2.4.1	<i>Beam Element Stiffness Matrix</i>	29
2.4.2	<i>Beam Element Mass Matrix</i>	31
2.5	Beam with an Axial Force	31
2.6	Beam on an Elastic Foundation	32
2.7	General Remarks on the Discretisation Process	33
2.8	Alternative Derivation of Element Stiffness	33
2.9	Two-dimensional Elements: Plane Stress	35
2.10	Energy Approach and Plane Strain	38
2.10.1	<i>Thermoelasticity</i>	39
2.11	Plane Element Mass Matrix	40
2.12	Axisymmetric Stress and Strain	40
2.13	Three-dimensional Stress and Strain	42
2.14	Plate Bending Element	44
2.15	Summary of Element Equations for Solids	47
2.16	Flow of Fluids: Navier–Stokes Equations	47
2.17	Simplified Flow Equations	50
2.17.1	<i>Steady State</i>	51
2.17.2	<i>Transient State</i>	53
2.17.3	<i>Convection</i>	53
2.18	Further Coupled Equations: Biot Consolidation	54
2.19	Conclusions	56
	References	56
3	Programming Finite Element Computations	59
3.1	Introduction	59
3.2	Local Coordinates for Quadrilateral Elements	59
3.2.1	<i>Numerical Integration for Quadrilaterals</i>	61
3.2.2	<i>Analytical Integration for Quadrilaterals</i>	63
3.3	Local Coordinates for Triangular Elements	64
3.3.1	<i>Numerical Integration for Triangles</i>	65
3.3.2	<i>Analytical Integration for Triangles</i>	65
3.4	Multi-Element Assemblies	66
3.5	‘Element-by-Element’ Techniques	68
3.5.1	<i>Conjugate Gradient Method for Linear Equation Systems</i>	68
3.5.2	<i>Preconditioning</i>	69

3.5.3	<i>Unsymmetric Systems</i>	70
3.5.4	<i>Symmetric Non-Positive Definite Equations</i>	71
3.5.5	<i>Eigenvalue Systems</i>	71
3.6	<i>Incorporation of Boundary Conditions</i>	72
3.6.1	<i>Convection Boundary Conditions</i>	74
3.7	<i>Programming using Building Blocks</i>	75
3.7.1	<i>Black Box Routines</i>	76
3.7.2	<i>Special Purpose Routines</i>	77
3.7.3	<i>Plane Elastic Analysis using Quadrilateral Elements</i>	77
3.7.4	<i>Plane Elastic Analysis using Triangular Elements</i>	81
3.7.5	<i>Axisymmetric Strain of Elastic Solids</i>	82
3.7.6	<i>Plane Steady Laminar Fluid Flow</i>	83
3.7.7	<i>Mass Matrix Formation</i>	83
3.7.8	<i>Higher-Order 2D Elements</i>	84
3.7.9	<i>Three-Dimensional Elements</i>	86
3.7.10	<i>Assembly of Elements</i>	90
3.8	<i>Solution of Equilibrium Equations</i>	95
3.9	<i>Evaluation of Eigenvalues and Eigenvectors</i>	96
3.9.1	<i>Jacobi Algorithm</i>	96
3.9.2	<i>Lanczos and Arnoldi Algorithms</i>	98
3.10	<i>Solution of First-Order Time-Dependent Problems</i>	99
3.11	<i>Solution of Coupled Navier–Stokes Problems</i>	103
3.12	<i>Solution of Coupled Transient Problems</i>	104
3.12.1	<i>Absolute Load Version</i>	105
3.12.2	<i>Incremental Load Version</i>	106
3.13	<i>Solution of Second-Order Time-Dependent Problems</i>	106
3.13.1	<i>Modal Superposition</i>	107
3.13.2	<i>Newmark or Crank–Nicolson Method</i>	109
3.13.3	<i>Wilson’s Method</i>	110
3.13.4	<i>Complex Response</i>	111
3.13.5	<i>Explicit Methods and Other Storage-Saving Strategies</i>	112
	<i>References</i>	113
4	Static Equilibrium of Structures	115
4.1	<i>Introduction</i>	115
	Program 4.1 One-dimensional analysis of axially loaded elastic rods using 2-node rod elements	116
	Program 4.2 Analysis of elastic pin-jointed frames using 2-node rod elements in two or three dimensions	121
	Program 4.3 Analysis of elastic beams using 2-node beam elements (elastic foundation optional)	127
	Program 4.4 Analysis of elastic rigid-jointed frames using 2-node beam/rod elements in two or three dimensions	133
	Program 4.5 Analysis of elastic–plastic beams or frames using 2-node beam or beam/rod elements in one, two or three dimensions	141
	Program 4.6 Stability (buckling) analysis of elastic beams using 2-node beam elements (elastic foundation optional)	150

	Program 4.7 Analysis of plates using 4-node rectangular plate elements. Homogeneous material with identical elements. Mesh numbered in x - or y -direction	153
4.2	Conclusions	157
4.3	Glossary of Variable Names	157
4.4	Exercises	159
	References	168
5	Static Equilibrium of Linear Elastic Solids	169
5.1	Introduction	169
	Program 5.1 Plane or axisymmetric strain analysis of a rectangular elastic solid using 3-, 6-, 10- or 15-node right-angled triangles or 4-, 8- or 9-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction	170
	Program 5.2 Non-axisymmetric analysis of a rectangular axisymmetric elastic solid using 8-node rectangular quadrilaterals. Mesh numbered in r - or z -direction	184
	Program 5.3 Three-dimensional analysis of a cuboidal elastic solid using 8-, 14- or 20-node brick hexahedra. Mesh numbered in xz -planes then in the y -direction	191
	Program 5.4 General 2D (plane strain) or 3D analysis of elastic solids. Gravity loading option	196
	Program 5.5 Plane or axisymmetric thermoelastic analysis of an elastic solid using 3-, 6-, 10- or 15-node right-angled triangles or 4-, 8- or 9-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction	205
	Program 5.6 Three-dimensional strain of a cuboidal elastic solid using 8-, 14- or 20-node brick hexahedra. Mesh numbered in xz -planes then in the y -direction. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver	209
	Program 5.7 Three-dimensional strain of a cuboidal elastic solid using 8-, 14- or 20-node brick hexahedra. Mesh numbered in xz -planes then in the y -direction. No global stiffness matrix. Diagonally preconditioned conjugate gradient solver. Optimised maths library, ABAQUS UMAT version	213
5.2	Glossary of Variable Names	221
5.3	Exercises	224
	References	232
6	Material Non-linearity	233
6.1	Introduction	233
6.2	Stress-strain Behaviour	235
6.3	Stress Invariants	236
6.4	Failure Criteria	238
	6.4.1 Von Mises	238
	6.4.2 Mohr-Coulomb and Tresca	239
6.5	Generation of Body Loads	240
6.6	Viscoplasticity	240

6.7	Initial Stress	242
6.8	Corners on the Failure and Potential Surfaces	243
	Program 6.1 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Viscoplastic strain method	244
	Program 6.2 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Viscoplastic strain method. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver	250
	Program 6.3 Plane-strain-bearing capacity analysis of an elastic–plastic (Mohr–Coulomb) material using 8-node rectangular quadrilaterals. Rigid smooth footing. Displacement control. Viscoplastic strain method	254
	Program 6.4 Plane-strain slope stability analysis of an elastic–plastic (Mohr–Coulomb) material using 8-node rectangular quadrilaterals. Gravity loading. Viscoplastic strain method	260
	Program 6.5 Plane-strain earth pressure analysis of an elastic–plastic (Mohr–Coulomb) material using 8-node rectangular quadrilaterals. Rigid smooth wall. Initial stress method	265
6.9	Elastoplastic Rate Integration	270
	6.9.1 <i>Forward Euler Method</i>	272
	6.9.2 <i>Backward Euler Method</i>	274
6.10	Tangent Stiffness Approaches	275
	6.10.1 <i>Inconsistent Tangent Matrix</i>	275
	6.10.2 <i>Consistent Tangent Matrix</i>	275
	6.10.3 <i>Convergence Criterion</i>	276
	Program 6.6 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Consistent tangent stiffness. Closest point projection method (CPPM)	277
	Program 6.7 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Consistent tangent stiffness. CPPM. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver	282
	Program 6.8 Plane-strain-bearing capacity analysis of an elastic–plastic (von Mises) material using 8-node rectangular quadrilaterals. Flexible smooth footing. Load control. Consistent tangent stiffness. Radial return method (RR) with ‘line search’	286
6.11	The Geotechnical Processes of Embanking and Excavation	289
	6.11.1 <i>Embanking</i>	289
	Program 6.9 Plane-strain construction of an elastic–plastic (Mohr–Coulomb) embankment in layers on a foundation using 8-node quadrilaterals. Viscoplastic strain method	290
	6.11.2 <i>Excavation</i>	294

	Program 6.10 Plane-strain construction of an elastic–plastic (Mohr–Coulomb) excavation in layers using 8-node quadrilaterals. Viscoplastic strain method	300
6.12	Undrained Analysis	305
	Program 6.11 Axisymmetric ‘undrained’ strain of an elastic–plastic (Mohr–Coulomb) solid using 8-node rectangular quadrilaterals. Viscoplastic strain method	308
	Program 6.12 Three-dimensional strain analysis of an elastic–plastic (Mohr–Coulomb) slope using 20-node hexahedra. Gravity loading. Viscoplastic strain method	313
	Program 6.13 Three-dimensional strain analysis of an elastic–plastic (Mohr–Coulomb) slope using 20-node hexahedra. Gravity loading. Viscoplastic strain method. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver	319
6.13	Glossary of Variable Names	322
6.14	Exercises	327
	References	331
7	Steady State Flow	333
7.1	Introduction	333
	Program 7.1 One-dimensional analysis of steady seepage using 2-node line elements	334
	Program 7.2 Plane or axisymmetric analysis of steady seepage using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction	337
	Program 7.3 Analysis of plane free surface flow using 4-node quadrilaterals. ‘Analytical’ form of element conductivity matrix	344
	Program 7.4 General two- (plane) or three-dimensional analysis of steady seepage	351
	Program 7.5 General two- (plane) or three-dimensional analysis of steady seepage. No global conductivity matrix assembly. Diagonally preconditioned conjugate gradient solver	355
7.2	Glossary of Variable Names	359
7.3	Exercises	361
	References	367
8	Transient Problems: First Order (Uncoupled)	369
8.1	Introduction	369
	Program 8.1 One-dimensional transient (consolidation) analysis using 2-node ‘line’ elements. Implicit time integration using the ‘theta’ method	370
	Program 8.2 One-dimensional transient (consolidation) analysis (settlement and excess pore pressure) using 2-node ‘line’ elements. Implicit time integration using the ‘theta’ method	373
	Program 8.3 One-dimensional consolidation analysis using 2-node ‘line’ elements. Explicit time integration. Element by element. Lumped mass	377

Program 8.4	Plane or axisymmetric transient (consolidation) analysis using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction. Implicit time integration using the 'theta' method	380
Program 8.5	Plane or axisymmetric transient (consolidation) analysis using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction. Implicit time integration using the 'theta' method. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver	388
Program 8.6	Plane or axisymmetric transient (consolidation) analysis using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction. Explicit time integration using the 'theta = 0' method	390
Program 8.7	Plane or axisymmetric transient (consolidation) analysis using 4-node rectangular quadrilaterals. Mesh numbered in $x(r)$ - or $y(z)$ -direction. 'theta' method using an element-by-element product algorithm	394
8.2	Comparison of Programs 8.4, 8.5, 8.6 and 8.7	397
Program 8.8	General two- (plane) or three-dimensional transient (consolidation) analysis. Implicit time integration using the 'theta' method	397
Program 8.9	Plane analysis of the diffusion-convection equation using 4-node rectangular quadrilaterals. Implicit time integration using the 'theta' method. Self-adjoint transformation	401
Program 8.10	Plane analysis of the diffusion-convection equation using 4-node rectangular quadrilaterals. Implicit time integration using the 'theta' method. Untransformed solution	405
Program 8.11	Plane or axisymmetric transient thermal conduction analysis using 4-node rectangular quadrilaterals. Implicit time integration using the 'theta' method. Option of convection and flux boundary conditions	410
8.3	Glossary of Variable Names	416
8.4	Exercises	419
	References	422
9	Coupled Problems	423
9.1	Introduction	423
Program 9.1	Analysis of the plane steady-state Navier-Stokes equation using 8-node rectangular quadrilaterals for velocities coupled to 4-node rectangular quadrilaterals for pressures. Mesh numbered in x -direction. Freedoms numbered in the order $u - p - v$	424
Program 9.2	Analysis of the plane steady-state Navier-Stokes equation using 8-node rectangular quadrilaterals for velocities coupled to 4-node rectangular quadrilaterals for pressures. Mesh numbered in x -direction. Freedoms numbered in the order $u - p - v$. Element-by-element solution using BiCGStab(l) with no preconditioning. No global matrix assembly	429

Program 9.3	One-dimensional coupled consolidation analysis of a Biot poroelastic solid using 2-node 'line' elements. Freedoms numbered in the order $v - u_w$	433
Program 9.4	Plane strain consolidation analysis of a Biot elastic solid using 8-node rectangular quadrilaterals for displacements coupled to 4-node rectangular quadrilaterals for pressures. Freedoms numbered in order $u - v - u_w$. Incremental load version	438
Program 9.5	Plane strain consolidation analysis of a Biot elastic solid using 8-node rectangular quadrilaterals for displacements coupled to 4-node rectangular quadrilaterals for pressures. Freedoms numbered in order $u - v - u_w$. Incremental load version. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver	445
Program 9.6	Plane strain consolidation analysis of a Biot poroelastic-plastic (Mohr-Coulomb) material using 8-node rectangular quadrilaterals for displacements coupled to 4-node rectangular quadrilaterals for pressures. Freedoms numbered in the order $u - v - u_w$. Viscoplastic strain method	448
9.2	Glossary of Variable Names	454
9.3	Exercises	459
	References	460
10	Eigenvalue Problems	461
10.1	Introduction	461
Program 10.1	Eigenvalue analysis of elastic beams using 2-node beam elements. Lumped mass	462
Program 10.2	Eigenvalue analysis of an elastic solid in plane strain using 4- or 8-node rectangular quadrilaterals. Lumped mass. Mesh numbered in y -direction	465
Program 10.3	Eigenvalue analysis of an elastic solid in plane strain using 4-node rectangular quadrilaterals. Lanczos method. Consistent mass. Mesh numbered in y -direction	469
Program 10.4	Eigenvalue analysis of an elastic solid in plane strain using 4-node rectangular quadrilaterals with ARPACK. Lumped mass. Element-by-element formulation. Mesh numbered in y -direction	474
10.2	Glossary of Variable Names	477
10.3	Exercises	480
	References	482
11	Forced Vibrations	483
11.1	Introduction	483
Program 11.1	Forced vibration analysis of elastic beams using 2-node beam elements. Consistent mass. Newmark time stepping	483
Program 11.2	Forced vibration analysis of an elastic solid in plane strain using 4- or 8-node rectangular quadrilaterals. Lumped mass. Mesh numbered in the y -direction. Modal superposition	489

Program 11.3	Forced vibration analysis of an elastic solid in plane strain using rectangular 8-node quadrilaterals. Lumped or consistent mass. Mesh numbered in the y-direction. Implicit time integration using the 'theta' method	493
Program 11.4	Forced vibration analysis of an elastic solid in plane strain using rectangular 8-node quadrilaterals. Lumped or consistent mass. Mesh numbered in the y-direction. Implicit time integration using Wilson's method	498
Program 11.5	Forced vibration of a rectangular elastic solid in plane strain using 8-node quadrilateral elements numbered in the y-direction. Lumped mass, complex response	501
Program 11.6	Forced vibration analysis of an elastic solid in plane strain using uniform size rectangular 4-node quadrilaterals. Mesh numbered in the y-direction. Lumped or consistent mass. Mixed explicit/ implicit time integration	504
Program 11.7	Forced vibration analysis of an elastic solid in plane strain using rectangular 8-node quadrilaterals. Lumped or consistent mass. Mesh numbered in the y-direction. Implicit time integration using the 'theta' method. No global matrix assembly. Diagonally preconditioned conjugate gradient solver	508
Program 11.8	Forced vibration analysis of an elastic-plastic (von Mises) solid in plane strain using rectangular 8-node quadrilateral elements. Lumped mass. Mesh numbered in the y-direction. Explicit time integration	512
11.2	Glossary of Variable Names	517
11.3	Exercises	521
	References	522
12	Parallel Processing of Finite Element Analyses	523
12.1	Introduction	523
12.2	Differences between Parallel and Serial Programs	525
12.2.1	<i>Parallel Libraries</i>	525
12.2.2	<i>Global Variables</i>	526
12.2.3	<i>MPI Library Routines</i>	526
12.2.4	<i>The _pp Appendage</i>	527
12.2.5	<i>Simple Test Problems</i>	527
12.2.6	<i>Reading and Writing</i>	530
12.2.7	<i>rest Instead of nf</i>	532
12.2.8	<i>Gathering and Scattering</i>	533
12.2.9	<i>Reindexing</i>	533
12.2.10	<i>Domain Composition</i>	533
12.2.11	<i>Third-party Mesh-partitioning Tools</i>	534
12.2.12	<i>Load Balancing</i>	535
Program 12.1	Three-dimensional analysis of an elastic solid. Compare Program 5.6	536
Program 12.2	Three-dimensional analysis of an elastoplastic (Mohr-Coulomb) solid. Compare Program 6.13	542
Program 12.3	Three-dimensional Laplacian flow. Compare Program 7.5	548

Program 12.4 Three-dimensional transient heat conduction—implicit analysis in time. Compare Program 8.5	553
Program 12.5 Three-dimensional transient flow—explicit analysis in time. Compare Program 8.6	562
Program 12.6 Three-dimensional steady-state Navier–Stokes analysis. Compare Program 9.2	565
Program 12.7 Three-dimensional analysis of Biot poro elastic solid. Incremental version. Compare Program 9.5	572
Program 12.8 Eigenvalue analysis of three-dimensional elastic solid. Compare Program 103	576
Program 12.9 Forced vibration analysis of a three-dimensional elastic solid. Implicit integration in time. Compare Program 11.7	581
Program 12.10 Forced vibration analysis of three-dimensional elastoplastic solid. Explicit integration in time. Compare Program 11.8	585
12.3 Graphics Processing Units	589
Program 12.11 Three-dimensional strain of an elastic solid using 8-, 14- or 20-node brick hexahedra. No global stiffness matrix assembly. Diagonally preconditioned conjugate gradient solver. GPU version. Compare Program 5.7	589
12.4 Cloud Computing	594
12.5 Conclusions	596
12.6 Glossary of Variable Names	597
References	602
Appendix A Equivalent Nodal Loads	605
Appendix B Shape Functions and Element Node Numbering	611
Appendix C Plastic Stress-Strain Matrices and Plastic Potential Derivatives	619
Appendix D <code>main</code> Library Subprograms	623
Appendix E <code>geom</code> Library Subroutines	635
Appendix F Parallel Library Subroutines	639
Appendix G External Subprograms	645
Author Index	649
Subject Index	653