

# Contents

|                      |   |               |
|----------------------|---|---------------|
| <b>Chapter 1</b>     | <b>Topics in Functional Analysis</b>                      | <b>1</b>      |
| 1.0                  | Introduction  | 1             |
| 1.1                  | Set Theory  | 2             |
| 1.2                  | Functions   | 5             |
| 1.3                  | Matrices  | 7             |
| 1.4                  | Solving Matrix Systems                                    | 9             |
| 1.5                  | Metric Spaces   | 18            |
| 1.6                  | Linear Spaces   | 22            |
| 1.7                  | Normed Linear Spaces                                      | 25            |
| 1.8                  | Approximations  | 31            |
| <br><b>Chapter 2</b> | <br><b>Integration Theory</b>                             | <br><b>37</b> |
| 2.0                  | Introduction  | 37            |
| 2.1                  | Reimann and Lebesgue Integrals: Step and Simple Functions | 37            |
| 2.2                  | Lebesgue Measure  | 38            |
| 2.3                  | Measurable Functions                                      | 40            |
| 2.4                  | The Lebesgue Integral                                     | 41            |
| 2.4.1                | Bounded Functions   | 42            |
| 2.4.2                | Unbounded Functions                                       | 44            |
| 2.5                  | Key Theorems in Integration Theory                        | 47            |
| 2.6                  | $L_p$ Spaces  | 49            |
| 2.6.1                | $m$ -Equivalent Functions                                 | 49            |
| 2.6.2                | The Space $L_p$   | 50            |
| 2.7                  | The Metric Space, $L_p$                                   | 51            |
| 2.8                  | Convergence of Sequences                                  | 51            |
| 2.8.1                | Common Modes of Convergence                               | 51            |
| 2.8.2                | Convergence in $L_p$                                      | 52            |
| 2.8.3                | Convergence in Measure ( <b>M</b> )                       | 52            |
| 2.8.4                | Almost Uniform Convergence ( <b>AU</b> )                  | 52            |
| 2.8.5                | Is the Approximation Converging?                          | 52            |
| 2.8.6                | Counterexamples   | 53            |
| 2.9                  | Capsulation   | 55            |
| <br><b>Chapter 3</b> | <br><b>Hilbert Space and Generalized Fourier Series</b>   | <br><b>57</b> |
| 3.0                  | Introduction  | 57            |
| 3.1                  | Inner Product and Hilbert Space                           | 58            |
| 3.2                  | Best Approximations in an Inner Product Space             | 62            |
| 3.3                  | Approximations in $L_2(E)$                                | 70            |
| 3.3.1                | Parseval's Identity                                       | 71            |
| 3.3.2                | Bessel's Inequality                                       | 71            |

|                  |  |            |
|------------------|--|------------|
| 3.4              | Vector Representations and Best Approximations   | 71         |
| 3.5              | Computer Program   | 82         |
| <b>Chapter 4</b> | <b>Linear Operators</b>  | <b>89</b>  |
| 4.0              | Introduction   | 89         |
| 4.1              | Linear Operator Theory   | 89         |
| 4.2              | Operator Norms   | 93         |
| 4.3              | Examples of Linear Operators in Engineering  | 97         |
| 4.4              | Superposition  | 101        |
| <b>Chapter 5</b> | <b>The Best Approximation Method</b>   | <b>104</b> |
| 5.0              | Introduction   | 104        |
| 5.1              | An Inner Product for the Solution of Linear Operator Equations                           | 104        |
| 5.2              | Definition of Inner Product and Norm   | 106        |
| 5.3              | Generalized Fourier Series   | 108        |
| 5.4              | Approximation Error Evaluation   | 117        |
| 5.5              | The Weighted Inner Product   | 124        |
| 5.6              | Considerations in Choosing Basis Functions   | 128        |
| 5.6.1            | Global Basis Elements  | 128        |
| 5.6.2            | Spline Basis Functions   | 129        |
| 5.6.3            | Mixed Basis Functions  | 133        |
| <b>Chapter 6</b> | <b>The Best Approximation Method: Applications</b>                                       | <b>134</b> |
| 6.0              | Introduction   | 134        |
| 6.1              | Sensitivity of Computational Results to Variation in the Inner Product Weighting Factor  | 134        |
| 6.2              | Solving Two-Dimensional Potential Problems   | 137        |
| 6.3              | Application to Other Linear Operators  | 146        |
| 6.4              | Computer Program: Two-Dimensional Potential Problems Using Real Variable Basis Functions | 150        |
| 6.4.1            | Introduction   | 150        |
| 6.4.2            | Input Data Description   | 152        |
| 6.4.3            | Computer Program Listing   | 154        |
| 6.5              | Application of Computer Program  | 166        |
| 6.5.1            | A Fourth Order Differential Equation   | 167        |
| <b>Chapter 7</b> | <b>Solving Potential Problems using the Best Approximation Method</b>                    | <b>170</b> |
| 7.0              | Introduction   | 170        |
| 7.1              | The Complex Variable Boundary Element Method   | 171        |
| 7.1.1            | Objectives   | 171        |
| 7.1.2            | Definition 7.1.1 (Working Space, $W_0$ )   | 171        |
| 7.1.3            | Definition 7.1.2 (the Function $  \omega  $ to $  \omega  _2$ )                          | 172        |

|        |  |     |
|--------|--|-----|
| 7.1.4  | Almost Everywhere (ae) Equality                            | 172 |
| 7.1.5  | Theorem (relationship of $  \omega  $ to $  \omega  _2$ )  | 172 |
| 7.1.6  | Theorem  | 173 |
| 7.1.7  | Theorem  | 173 |
| 7.2    | Mathematical Development                                   | 174 |
| 7.2.1  | Discussion: (A Note on Hardy Spaces)                       | 174 |
| 7.2.2  | Theorem (Boundary Integral Representation)                 | 174 |
| 7.2.3  | Almost Everywhere (ae) Equivalence                         | 174 |
| 7.2.4  | Theorem (Uniqueness of Zero Element in $W_0$ )             | 175 |
| 7.2.5  | Theorem ( $W_0$ is a Vector Space)                         | 175 |
| 7.2.6  | Theorem (Definition of the Inner-Product)                  | 176 |
| 7.2.7  | Theorem ( $W_0$ is an Inner-Product Space)                 | 176 |
| 7.2.8  | Theorem ( $  \omega  $ is a Norm on $W_0$ )                | 176 |
| 7.2.9  | Theorem  | 176 |
| 7.3    | The CVBEM and $W_0$  | 176 |
| 7.3.1  | Definition 7.3.1 (Angle Points)                            | 176 |
| 7.3.2  | Definition 7.3.2 (Boundary Element)                        | 177 |
| 7.3.3  | Theorem  | 177 |
| 7.3.4  | Definition 7.3.3 (Linear Basis Function)                   | 177 |
| 7.3.5  | Theorem  | 177 |
| 7.3.6  | Definition 7.3.4 (Global Trial Function)                   | 177 |
| 7.3.7  | Theorem  | 178 |
| 7.3.8  | Discussion   | 178 |
| 7.3.9  | Theorem  | 178 |
| 7.3.10 | Discussion   | 178 |
| 7.3.11 | Theorem (Linear Independence of Nodal Expansion Functions) | 180 |
| 7.3.12 | Discussion   | 181 |
| 7.3.13 | Theorem  | 181 |
| 7.3.14 | Theorem  | 182 |
| 7.3.15 | Discussion   | 182 |
| 7.4    | The Space $W_0^\wedge$                                     | 183 |
| 7.4.1  | Definition 7.4.1 ( $W_0^\wedge$ )                          | 183 |
| 7.4.2  | Theorem  | 183 |
| 7.4.3  | Theorem  | 183 |
| 7.4.4  | Discussion   | 184 |
| 7.4.5  | Theorem  | 184 |
| 7.4.6  | Theorem  | 185 |
| 7.4.7  | Discussion: Another Look at $W_0$                          | 185 |
| 7.5    | Applications   | 185 |
| 7.5.1  | Introduction   | 185 |
| 7.5.2  | Nodal Point Placement on $\Gamma$                          | 186 |
| 7.5.3  | Potential Flow-Field (Flow-Net) Development                | 186 |
| 7.5.4  | Approximate Boundary Development                           | 186 |
| 7.5.5  | Application Problems                                       | 187 |

|                   |   |            |
|-------------------|---|------------|
| 7.6               | Computer Program: Two-Dimensional Potential Problems using Analytic Basis Functions (CVBEM) | 187        |
| 7.6.1             | Introduction  | 187        |
| 7.6.2             | CVBEM1 Program Listing  | 191        |
| 7.6.3             | Input Variable Description for CVBEM1   | 203        |
| 7.6.4             | CVBEM2 Program Listing  | 204        |
| 7.7               | Modelling Groundwater Contaminant Transport   | 213        |
| 7.7.1             | Application 1A  | 214        |
| 7.7.2             | Application 1B  | 214        |
| 7.7.3             | Application 2A  | 214        |
| 7.7.4             | Application 2B  | 214        |
| 7.8               | Three Dimensional Potential Problems  | 217        |
| 7.8.1             | Approximation Error Evaluation - Approximate Boundary Method                                | 217        |
| 7.8.2             | Computer Implementation   | 218        |
| 7.8.3             | Application   | 219        |
| 7.8.4             | Trial Functions   | 219        |
| 7.8.5             | Constructing the Approximate Boundary, $\hat{\Gamma}$                                       | 221        |
| <b>Chapter 8</b>  | <b>Applications to Linear Operator Equations</b>  | <b>222</b> |
| 8.0               | Introduction  | 222        |
| 8.1               | Data Fit Analysis   | 222        |
| 8.2               | Ordinary Differential Equations   | 223        |
| 8.3               | Best Approximation of Function  | 226        |
| 8.4               | Matrix Systems  | 228        |
| 8.5               | Linear Partial Differential Equations   | 230        |
| 8.6               | Linear Integral Equations   | 233        |
| 8.6.1             | An Inverse Problem  | 234        |
| 8.6.2             | Best Approximation of the Transfer Function in a Linear Space                               | 236        |
| <b>References</b> |   | <b>238</b> |
| <b>Appendix A</b> | <b>Derivation of CVBEM Approximation Function</b>   | <b>239</b> |
| <b>Appendix B</b> | <b>Convergence of CVBEM Approximator</b>  | <b>243</b> |
| <b>Appendix C</b> | <b>The Approximate Boundary for Error Analysis</b>  | <b>245</b> |
| <b>Index</b>      |   | <b>249</b> |