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PART I

FUNDAMENTALS

1 Ordinary Differential Equations 3

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|------|-----|--|----|
| (E) | 1.1 | Ordinary Differential Equations (definitions; introductory examples) | 3 |
| (E) | 1.2 | Initial-Value and Boundary-Value Problems (definitions; comparison of local and global analysis; examples of initial-value problems) | 5 |
| (TE) | 1.3 | Theory of Homogeneous Linear Equations (linear dependence and independence; Wronskians; well-posed and ill-posed initial-value and boundary-value problems) | 7 |
| (E) | 1.4 | Solutions of Homogeneous Linear Equations (how to solve constant-coefficient, equidimensional, and exact equations; reduction of order) | 11 |
| (E) | 1.5 | Inhomogeneous Linear Equations (first-order equations; variation of parameters; Green's functions; delta function; reduction of order; method of undetermined coefficients) | 14 |
| (E) | 1.6 | First-Order Nonlinear Differential Equations (methods for solving Bernoulli, Riccati, and exact equations; factoring; integrating factors; substitutions) | 20 |
| (I) | 1.7 | Higher-Order Nonlinear Differential Equations (methods to reduce the order of autonomous, equidimensional, and scale-invariant equations) | 24 |

† Each section is labeled according to difficulty: **(E)** = easy, **(I)** = intermediate, **(D)** = difficult. A section labeled **(T)** indicates that the material has a theoretical rather than an applied emphasis.

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|------|---|-----------|---|
| (E) | 1.8 Eigenvalue Problems | 27 | (examples of eigenvalue problems on finite and infinite domains) |
| (TE) | 1.9 Differential Equations in the Complex Plane | 29 | (comparison of real and complex differential equations) |
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| | 2 Difference Equations | 36 | |
| (E) | 2.1 The Calculus of Differences | 36 | (definitions; parallels between derivatives and differences, integrals, and sums) |
| (E) | 2.2 Elementary Difference Equations | 37 | (examples of simple linear and nonlinear difference equations; gamma function; general first-order linear homogeneous and inhomogeneous equations) |
| (I) | 2.3 Homogeneous Linear Difference Equations | 40 | (constant-coefficient equations; linear dependence and independence; Wronskians; initial-value and boundary-value problems; reduction of order; Euler equations; generating functions; eigenvalue problems) |
| (I) | 2.4 Inhomogeneous Linear Difference Equations | 49 | (variation of parameters; reduction of order; method of undetermined coefficients) |
| (E) | 2.5 Nonlinear Difference Equations | 53 | (elementary examples) |
| | Problems for Chapter 2 | 53 | |

PART II

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| (E) | 3.2 Local Behavior Near Ordinary Points of Homogeneous Linear Equations | 66 | (Taylor series solution of first- and second-order equations; Airy equation) |
| (I) | 3.3 Local Series Expansions About Regular Singular Points of Homogeneous Linear Equations | 68 | (methods of Fuchs and Frobenius; modified Bessel equation) |
| (E) | 3.4 Local Behavior at Irregular Singular Points of Homogeneous Linear Equations | 76 | (failure of Taylor and Frobenius series; asymptotic relations; controlling factor and leading behavior; method of dominant balance; asymptotic series expansion of solutions at irregular singular points) |

- (E) 3.5 Irregular Singular Point at Infinity 88
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- (E) 3.6 Local Analysis of Inhomogeneous Linear Equations 103
(illustrative examples)
- (TI) 3.7 Asymptotic Relations 107
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- (TD) 3.8 Asymptotic Series 118
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- (E) 5.1 Introductory Comments 205
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- (I) 5.2 Ordinary and Regular Singular Points of Linear Difference Equations 206
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