

CONTENTS

PRINCIPAL NOTATION	xxi
1 INTRODUCTION TO FINITE ELEMENT METHOD	1
1.1 Basic concept	1
1.2 Historical background	1
1.3 General applicability of the method	3
1.3.1 One-dimensional heat transfer	3
1.3.2 One-dimensional fluid flow	5
1.3.3 Solid bar under axial load	5
1.4 Engineering applications of the finite element method	6
1.5 General description of the finite element method	8
1.6 Comparison of finite element method with other methods of analysis	21
1.6.1 Derivation of the equation of motion for the vibration of a beam	21
1.6.2 Exact analytical solution (separation of variables technique)	23
1.6.3 Approximate analytical solution (Rayleigh's method)	24
1.6.4 Approximate analytical solution (Galerkin method)	26
1.6.5 Finite difference method of numerical solution	28
1.6.6 Finite element method of numerical solution (displacement method)	30
1.7 Finite element program packages	32
References	35
Problems	41

2	SOLUTION OF FINITE ELEMENT EQUATIONS	43
2.1	Introduction	43
2.2	Solution of equilibrium problems	45
2.2.1	Gaussian elimination method	46
(i)	Generalization of the method	47
(ii)	Computer implementation of Gaussian elimination method (GAUSS)	48
2.2.2	Choleski method	51
(i)	Decomposition of [A] into lower and upper triangular matrices	51
(ii)	Solution of equations	52
(iii)	Choleski decomposition of symmetric matrices	52
(iv)	Inverse of a symmetric matrix	53
(v)	Computer implementation of the Choleski method (DECOMP and SOLVE)	54
2.2.3	Other methods	57
2.3	Solution of eigenvalue problems	57
2.3.1	Jacobi method	59
(i)	Method	60
(ii)	Computer implementation of the Jacobi method (JACOBI)	60
2.3.2	Power method	63
(i)	Computing the largest eigenvalue by the power method	63
(ii)	Computing the smallest eigenvalue by the power method	65
(iii)	Computing intermediate eigenvalues	65
2.3.3	Rayleigh-Ritz subspace iteration method	68
(i)	Algorithm	68
(ii)	Computer implementation of subspace iteration method (SUSPIT)	69
2.3.4	Other methods	78
2.4	Solution of propagation problems	79
2.4.1	Numerical solution of Eq. (2.56)	80
(i)	Solution of a set of first order differential equations	81
(ii)	Computer implementation of Runge-Kutta method (RUNGE)	81
2.4.2	Numerical solution of Eq. (2.58)	85
(i)	Direct integration methods	85
(ii)	Mode superposition methods	87

(iii) Solution of a general second order differential equation	88
(iv) Computer implementation of mode superposition method (MODAL)	90
2.5 Parallel processing in finite element analysis	93
References	94
Problems	96
3 GENERAL PROCEDURE OF FINITE ELEMENT METHOD	101
3.1 Discretization of the domain	101
3.1.1 Basic element shapes	101
3.1.2 Discretization process	105
(i) Type of elements	105
(ii) Size of elements	108
(iii) Location of nodes	110
(iv) Number of elements	110
(v) Simplifications afforded by the physical configuration of the body	111
(vi) Finite representation of infinite bodies	111
(vii) Node numbering scheme	113
(viii) Automatic mesh generation	116
3.2 Interpolation polynomials	116
3.2.1 Polynomial form of interpolation functions	117
(i) Simplex, complex and multiplex elements	119
(ii) Interpolation polynomial in terms of nodal degrees of freedom	120
3.2.2 Selection of the order of the interpolation polynomial	121
3.2.3 Convergence requirements	123
3.2.4 Linear interpolation polynomials in terms of global coordinates	125
(i) One-dimensional simplex element	125
(ii) Two-dimensional simplex element	129
(iii) Three-dimensional simplex element	131
(iv) Interpolation polynomials for vector quantities	133
3.2.5 Linear interpolation polynomials in terms of local coordinates	136
(i) One-dimensional element	138
(ii) Two-dimensional (triangular) element	140
(iii) Three-dimensional (tetrahedron) element	143
3.3 Formulation of element characteristic matrices and vectors	146
3.3.1 Direct approach	147
(i) Bar element under axial load	147
(ii) Line element for heat flow	148

xii	Contents	
	(iii) Line element for fluid flow	150
	(iv) Line element for current flow	151
	(v) Triangular element under plane strain	152
3.3.2	Variational approach	154
	(i) Specification of continuum problems	155
	(ii) Approximate methods of solving continuum problems	155
	(iii) Calculus of variations	155
	(iv) Advantages of variational formulation	160
	(v) Solution of equilibrium problems using variational (Rayleigh-Ritz) method	160
	(vi) Solution of eigenvalue problems using variational (Rayleigh-Ritz) method	164
	(vii) Solution of propagation problems using variational (Rayleigh-Ritz) method	165
	(viii) Equivalence of finite element method and variational (Rayleigh-Ritz) method	165
	(ix) Derivation of finite element equations using variational (Rayleigh-Ritz) approach	166
3.3.3	Weighted residual approach	172
	(i) Solution of equilibrium problems using weighted residual method	173
	(ii) Solution of eigenvalue problems using weighted residual method	177
	(iii) Solution of propagation problems using weighted residual method	178
	(iv) Derivation of finite element equations using weighted residual (Galerkin) approach	179
3.3.4	Coordinate transformation	182
3.4	Assembly of element matrices and vectors and derivation of system equations	183
3.4.1	Assemblage of element equations	183
3.4.2	Computer implementation of the assembly procedure	185
3.4.3	Incorporation of the boundary conditions	194
3.4.4	Incorporation of boundary conditions in the computer program	196
3.5	Solution of finite element (system) equations	197
3.6	Computation of element resultants	198
	References	198
	Problems	200
4	HIGHER ORDER AND ISOPARAMETRIC ELEMENT FORMULATIONS	205

4.1	Introduction	205
4.2	Higher order one-dimensional element	206
4.2.1	Quadratic element	206
4.2.2	Cubic element	207
4.3	Higher order elements in terms of natural coordinates	208
4.3.1	One-dimensional element	208
4.3.2	Two-dimensional element (triangular element)	210
4.3.3	Derivation of nodal interpolation functions	212
4.3.4	Three-dimensional element (tetrahedron element)	215
4.3.5	Two-dimensional element (quadrilateral element)	217
4.3.6	Three-dimensional element (hexahedron element)	221
4.4	Higher order elements in terms of classical interpolation polynomials	225
4.4.1	Classical interpolation functions	225
(i)	Lagrange interpolation functions for n stations	225
(ii)	General two-station interpolation functions	227
(iii)	Zeroth order Hermite interpolation function	228
(iv)	First order Hermite interpolation function	230
(v)	Second order Hermite interpolation function	232
4.4.2	One-dimensional elements	233
(i)	Linear element	233
(ii)	Quadratic element	233
(iii)	Cubic element	233
4.4.3	Two-dimensional elements: Rectangular elements	234
(i)	Using Lagrange interpolation polynomials	234
(ii)	Using Hermite interpolation polynomials	235
4.5	Continuity conditions	237
4.6	Comparative study of elements	239
4.7	Isoparametric elements	240
4.7.1	Definitions	240
4.7.2	Shape functions in coordinate transformation	241
4.7.3	Curved-sided elements	243
4.7.4	Derivation of element equations	246
4.8	Numerical integration	248
4.8.1	In one-dimension	248
4.8.2	In two-dimensions	250
(i)	In rectangular regions	250
(ii)	In triangular regions	251

4.8.3 In three-dimensions	252
(i) In rectangular prism type regions	252
(ii) In tetrahedral regions	252
References	253
Problems	254
5 SOLID AND STRUCTURAL MECHANICS	257
5.1 Introduction	257
5.2 Basic equations of solid mechanics	258
5.2.1 Introduction	258
5.2.2 External equilibrium equations	259
5.2.3 Equations of internal equilibrium	259
5.2.4 Stress-strain relations (Constitutive relations)	261
(i) Three-dimensional case	261
(ii) Two-dimensional case (plane stress)	262
(iii) Two-dimensional case (plane strain)	263
(iv) One-dimensional case	265
(v) Axisymmetric case	265
5.2.5 Strain-displacement relations	266
5.2.6 Boundary conditions	268
5.2.7 Compatibility equations	270
5.2.8 Stress-strain relations for anisotropic materials	271
5.2.9 Formulations of solid and structural mechanics	272
STATIC ANALYSIS	
5.3 Formulation of equilibrium equations	278
5.4 Analysis of trusses and frames	283
5.4.1 Space truss element	283
5.4.2 Space frame element	291
(i) Axial displacements	291
(ii) Torsional displacements	294
(iii) Bending displacements in the plane xy	296
(iv) Bending displacements in the plane xz	297
5.4.3 Planar frame element	304
5.4.4 Beam element	306
5.4.5 Computer program for frame analysis (FRAME)	307
5.5 Analysis of plates	315

Contents

xv

5.5.1	Introduction	315
5.5.2	Triangular membrane element	315
5.5.3	Numerical results with membrane element	323
	(i) A plate under tension	323
	(ii) Circular hole in a tension plate	325
	(iii) Cantilevered box beam	328
5.5.4	Computer program for plates under inplane loads (CST)	329
5.5.5	Bending behaviour of plates	335
5.5.6	Triangular plate bending element	340
5.5.7	Numerical results with bending elements	345
5.5.8	Analysis of three-dimensional structures using plate elements	348
5.5.9	Computer program for the analysis of three-dimensional structures using plate elements (PLATE)	352
5.6	Analysis of three-dimensional problems	352
5.6.1	Introduction	352
5.6.2	Tetrahedron element	352
5.6.3	Hexahedron element	355
5.6.4	Numerical results	360
5.7	Analysis of solids of revolution	360
5.7.1	Introduction	360
5.7.2	Formulation of elemental equations for an axisymmetric ring element	361
5.7.3	Numerical results	365
5.7.4	Computer program (STRESS)	366
DYNAMIC ANALYSIS		
5.8	Dynamic equations of motion	374
5.9	Consistent and lumped mass matrices	377
5.10	Consistent mass matrices in global coordinate system	378
5.10.1	Consistent mass matrix of a pin-jointed (space truss) element	379
5.10.2	Consistent mass matrix of a frame element	380
5.10.3	Consistent mass matrix of a triangular membrane element	382
5.10.4	Consistent mass matrix of a triangular bending element	383
5.10.5	Consistent mass matrix of a tetrahedron element	384
5.11	Free vibration analysis	385
5.12	Computer program for eigenvalue analysis of three-dimensional structures (PLATE)	393

5.13 Condensation of the eigenvalue problem (eigenvalue economizer)	407
(i) Natural frequencies of a square cantilever plate	410
(ii) Natural frequencies of a cantilevered box beam	411
5.14 Dynamic response calculations using finite element method	412
5.14.1 Uncoupling the equations of motion of an undamped system	413
5.14.2 Uncoupling the equations of motion of a damped system	414
5.14.3 Solution of a general second order differential equation	415
5.15 Nonconservative stability and flutter problems	422
5.16 Substructures method	423
References	424
Problems	426
6 HEAT TRANSFER	432
6.1 Introduction	432
6.2 Basic equations of heat transfer	433
6.2.1 Energy balance equation	433
6.2.2 Rate equations	433
(i) For conduction	433
(ii) For convection	434
(iii) For radiation	434
(iv) Energy generated in a solid	434
(v) Energy stored in a solid	435
6.2.3 Governing differential equation for heat conduction in three-dimensional bodies	435
6.2.4 Statement of the problem in differential equation form	439
6.3 Derivation of finite element equations	439
6.3.1 Variational approach	439
6.3.2 Galerkin approach	442
6.4 One-dimensional heat transfer	445
6.4.1 Straight uniform fin analysis	445
Computer program (HEAT1)	453
6.4.2 Tapered fin analysis	455
6.4.3 Straight uniform fin analysis using quadratic elements	459
6.5 Two-dimensional heat transfer	462
Computer program (HEAT2)	478
6.6 Axisymmetric heat transfer	482
Computer program (HEATAX)	491
6.7 Three-dimensional heat transfer	496

Contents**xvii**

6.8 Unsteady state heat transfer problems	501
6.8.1 Derivation of element capacitance matrices	501
(i) For one-dimensional problems	501
(ii) For two-dimensional problems	503
(iii) For axisymmetric problems	503
(iv) For three-dimensional problems	504
6.8.2 Finite difference solution in the time domain	507
6.9 Heat transfer problems with radiation	509
Computer program (RADIAT)	515
References	518
Problems	518
7 FLUID MECHANICS	521
7.1 Introduction	521
7.2 Basic equations of fluid mechanics	522
7.2.1 Definitions	522
7.2.2 Flow field	522
7.2.3 Continuity equation	523
7.2.4 Equations of motion or momentum equations	524
(i) State of stress in a fluid	524
(ii) Relation between stress and rate of strain for Newtonian fluids	525
(iii) Equations of motion	527
7.2.5 Energy equation	529
7.2.6 State and viscosity equations	531
7.2.7 Solution procedure	531
7.2.8 Inviscid fluid flow	531
7.2.9 Irrotational flow	532
7.2.10 Velocity potential	534
7.2.11 Stream function	534
7.2.12 Bernoulli equation	536
7.3 Inviscid incompressible flows	537
7.3.1 Potential function formulation	538
(i) Differential equation form	538
(ii) Variational form	538
7.3.2 Stream function formulation	547
(i) Differential equation form	547
(ii) Variational form	547
7.3.3 Computer program (PHIFLO)	549

7.4	Flow in porous media	552
7.4.1	Governing equations	553
7.4.2	Finite element solution	554
7.4.3	Steady state unconfined flow through a dam	556
7.4.4	Steady state flow towards a well	557
7.5	Wave motion of a shallow basin	558
7.5.1	Equation of motion	558
7.5.2	Boundary and initial conditions	560
7.5.3	Finite element solution of Eq. (7.133) using Galerkin approach	560
7.5.4	Eigenvalue solution	564
7.5.5	Solution of Eq. (7.161) by mode superposition method	565
7.6	Incompressible viscous flow	566
7.6.1	Statement of the problem	566
7.6.2	Stream function formulation (using variational approach)	566
7.6.3	Velocity-pressure formulation (using Galerkin approach)	572
7.6.4	Stream function-vorticity formulation	577
	(i) Governing equations	577
	(ii) Finite element solution (using variational approach)	577
7.7	Flow of non-Newtonian fluids	579
7.7.1	Governing equations	579
	(i) Flow curve characteristic	579
	(ii) Equation of motion	581
7.7.2	Finite element equations using Galerkin method	581
7.7.3	Solution procedure	582
7.8	Other developments	585
	References	585
	Problems	587
8	ADDITIONAL APPLICATIONS AND GENERALIZATION OF THE FINITE ELEMENT METHOD	588
8.1	Introduction	588
8.2	Steady state field problems	589
8.3	Transient field problems	592
8.4	Space-time finite elements	594
8.5	Solution of Poisson equation	595
8.5.1	Derivation of the governing equation for the torsion problem	595
8.5.2	Finite element solution	597
8.5.3	Computer program (TORSON)	603

	Contents	xix
8.6 Solution of Helmholtz equation	605	
8.7 Solution of Reynolds equation	611	
8.7.1 Hydrodynamic lubrication problem	611	
8.7.2 Finite element equations	612	
8.8 Least squares finite element approach	616	
8.8.1 Solution of a general linear partial differential equation	616	
8.8.2 Solution of unsteady gas dynamic equations	620	
8.9 Equilibrium, mixed and hybrid elements	625	
8.10 Penalty finite element method	626	
8.11 Miscellaneous applications	627	
References	629	
Problems	634	
APPENDIX A: GREEN-GAUSS THEOREM (Integration by parts in two and three dimensions)	635	
APPENDIX B: FINITE ELEMENT PROGRAMS	637	
INDEX	639	