CONTENTS

PREFACE xv

NOMENCLATURE xvii

PART I: THE MATHEMATICAL MODELS FOR FLUID FLOW SIMULATIONS AT VARIOUS LEVELS OF APPROXIMATION 1

Introduction 1

Chapter 1 The Basic Equations of Fluid Dynamics 8

1.1 General Form of a Conservation Law 9
1.1.1 Scalar conservation law 9
1.1.2 Vector conservation law 11
1.2 The Equation of Mass Conservation 12
1.3 The Conservation Law of Momentum or Equation of Motion 14
1.4 Rotating Frame of Reference 16
1.5 The Conservation Equation for Energy 18
1.5.1 Conservative formulation of the energy equation 19
1.5.2 The equations for internal energy and entropy 20
1.5.3 Energy equation in a relative system 21
1.5.4 Crocco's form of the equations of motion 21

Chapter 2 The Dynamic Levels of Approximation 26

2.1 The Navier–Stokes Equations 29
2.1.1 Perfect gas model 31
2.1.2 Incompressible fluid model 44
2.2 The Reynolds-Averaged Navier–Stokes Equations 49
2.3 The Thin Shear Layer (TSL) Approximation 63
2.4 The Parabolized Navier–Stokes Approximation 70
2.5 The Boundary Layer Approximation 75
2.6 The Distributed Loss Model 81
2.7 The Inviscid Flow Model—Euler Equations 87
2.7.1 The properties of discontinuous solutions 88
2.8 Steady Inviscid Rotational Flows—Clebsch Representation 100
2.9 The Potential Flow Model 108
2.9.1 Irrotational flow with circulation—Kutta–Joukowski condition 109
2.9.2 The limitations of the potential flow model for transonic flows 110

2.9.3 The non-uniqueness of transonic potential models 120
2.9.4 The small-disturbance approximation of the potential equation 126
2.9.5 Linearized potential flows—singularity methods 127
2.10 Summary 128

Chapter 3 The Mathematical Nature of the Flow Equations and their Boundary Conditions 133

3.1 Introduction 133
3.2 The concept of characteristic surfaces and wave-like solutions 135
3.2.1 Partial differential equation of second order 135
3.2.2 Wave front or characteristic surfaces 137
3.2.3 General definition 139
3.2.4 Domain of dependence—zone of influence 143
3.3 Alternative Definition—Compatibility Relations 145
3.3.1 Compatibility relations 146
3.4 Time-like Variables 148
3.4.1 Plane wave solutions with time-like variable 149
3.4.2 Non-linear wave solutions and time-like variable 151
3.5 Initial and Boundary Conditions 152

PART II: BASIC DISCRETIZATION TECHNIQUES 161

Chapter 4 The Finite Difference Method 167

4.1 The Basics of Finite Difference Methods 167
4.1.1 The properties of difference formulas 168
4.1.2 Difference formulas with an arbitrary number of points 169
4.2 General Methods for Finite Difference Formulas 171
4.2.1 Generation of difference formulas for first derivatives 173
4.2.2 Higher-order derivatives 176
4.3 Implicit Finite Difference Formulas 180
4.3.1 General derivation of implicit finite difference formulas for first and second derivatives 183
4.4 Multi-dimensional Finite Difference Formulas 186
4.4.1 Difference schemes for the Laplace operator 187
4.4.2 Mixed derivatives 191
4.5 Finite Difference Formulas on Non-Uniform Cartesian Meshes 195

Chapter 5 The Finite Element Method 201

5.1 The Nature of the Finite Element Approximation 202
5.1.1 Finite element definition of the space 202
5.1.2 Finite element definition of interpolation functions 203
5.1.3 Finite element definition of the equation discretization—integral formulation 204
5.2 The Finite Element Interpolation Functions 205
5.2.1 One-dimensional elements 205
5.2.2 Two-dimensional elements 211
5.2.3 Three-dimensional elements 215

5.3 Integral Formulation: The Method of Weighted Residuals or Weak Formulation 216
5.3.1 The Galerkin method 218
5.3.2 Finite element Galerkin method for a conservation law 221
5.3.3 Subdomain collocation—finite volume method 223
5.4 Practical Computational Techniques 225
5.4.1 General mapping to local co-ordinates 226
5.4.2 Numerical integration techniques 227

Chapter 6 Finite Volume Method and Conservative Discretizations 237

6.1 The Conservative Discretization 237
6.2 The Finite Volume Method 241
6.2.1 Two-dimensional finite volume method 246
6.2.2 General integration formulas for finite volumes 253
6.2.3 Three-dimensional finite volume method 256

PART III: THE ANALYSIS OF NUMERICAL SCHEMES 265

Chapter 7 The Concepts of Consistency, Stability and Convergence 267

7.1 Model Equations 267
7.1.1 One-dimensional simplified models 267
7.1.2 Two-dimensional simplified models 269
7.2 Basic Definitions: Consistency, Stability, Convergence 270
7.2.1 Consistency 276
7.2.2 Stability 278
7.2.3 Convergence 281

Chapter 8 The Von Neumann Method for Stability Analysis 283

8.1 Fourier Decomposition of the Error 284
8.1.1 Amplification factor 286
8.1.2 Comment on the CFL condition 288
8.2 General Formulation of Von Neumann's Method: System of Equations 289
8.2.1 Matrix and operator formulation 290
8.2.2 The general Von Neumann stability condition 295
8.3 The Spectral Analysis of Numerical Errors 301
8.3.1 Error analysis for parabolic problems 303
8.3.2 Error analysis for hyperbolic problems 305
8.3.3 Extension to three-level schemes 311
8.3.4 A comparison of different schemes for the linear convection equation 314
8.3.5 The numerical group velocity 317
8.4 Multi-dimensional Von Neumann Analysis 319
8.4.1 Parabolic equations 320
8.4.2 The two-dimensional convection equation 321
8.5 Stability Conditions for Non-Linear Problems 322
8.5.1 Non-constant coefficients 322
8.5.2 Dissipative schemes (Kreiss, 1964) 324
8.5.3 Non-linear problems 325

8.6 Some General Methods for the Determination of Von Neumann Stability Conditions 329
8.6.1 One-dimensional, two-level, three-point schemes 329
8.6.2 Multi-dimensional space-centred, convection–diffusion equation 331
8.6.3 General multi-level, multi-dimensional schemes 335

Chapter 9 The Method of the Equivalent Differential Equation for the Analysis of Stability 342

9.1 Stability Analysis for Parabolic Problems 343
9.2 Stability and Accuracy Analysis for Hyperbolic Problems 345
9.2.1 General formulation of the equivalent differential equation for linear hyberbolic problems 347
9.2.2 Error estimations for two-level explicit schemes 353
9.2.3 Stability analysis for two-level explicit schemes 355
9.3 The Generation of New Algorithms with a Prescribed Order of Accuracy 356
9.4 The Equivalent Differential Equation for Non-Linear Hyperbolic Problems 360

Chapter 10 The Matrix Method for Stability Analysis 370

10.1 Principle of the Matrix Method—Analysis of the Space Discretization 370
10.1.1 Amplification factors and stability criteria 374
10.2 The Spectra of Space-Discretized Operators 380
10.2.1 The spectrum for the diffusion equation $u_t = \alpha u_{xx}$ 380
10.2.2 The spectrum for the convection equation $u_t + au_x = 0$ 386
10.3 The Stability of Time-Integration Schemes 391
10.3.1 Euler explicit scheme 391
10.3.2 Leapfrog method 393
10.3.3 Euler implicit (backward) scheme 394
10.3.4 Stability region in the complex Ω plane 394
10.3.5 A realistic example (Eriksson and Rizzi, 1985) 396
10.4 Evaluation of Stability Criteria 400
10.4.1 The stability analysis of the convection–diffusion equation 403
10.5 Normal Mode Representation 408
10.5.1 Exact solutions of a space difference scheme 409
10.5.2 Spatial propagation of errors in time-dependent schemes 412

PART IV: THE RESOLUTION OF DISCRETIZED EQUATIONS 421

Chapter 11 Integration Methods for Systems of Ordinary Differential Equations 423

11.1 Linear Multi-step Methods 424
11.2 Predictor–Corrector Schemes 431
11.3 Linearization Methods for Non-Linear Implicit Schemes 435
11.4 Implicit Schemes for Multi-dimensional Problems: Alternating Direction Implicit (ADI) Methods 437
11.4.1 Two-dimensional diffusion equation 440
11.4.2 ADI method for the convection equation 442
11.5 The Runge–Kutta Schemes 445

Chapter 12 Iterative Methods for the Resolution of Algebraic Systems 456

12.1 Basic Iterative Methods 457
12.1.1 Poisson's equation on a Cartesian, two-dimensional mesh 457
12.1.2 Point Jacobi method—Point Gauss–Seidel method 460
12.1.3 Convergence analysis of iterative schemes 462
12.1.4 Eigenvalue analysis of an iterative method 465
12.1.5 Fourier analysis of an iterative method 468
12.2 Overrelaxation Methods 469
12.2.1 Jacobi overrelaxation 469
12.2.2 Gauss–Seidel overrelaxation—successive overrelaxation (SOR) 471
12.2.3 Symmetric successive overrelaxation (SSOR) 473
12.2.4 Successive line overrelaxation methods (SLOR) 474
12.3 Preconditioning Techniques 478
12.3.1 Richardson method 479
12.3.2 Alternating direction implicit (ADI) method 481
12.3.3 Other preconditioning techniques 482
12.4 Non-linear Problems 484
12.5 The Differential Equation Representation of a Relaxation Scheme 486
12.6 The Multi-grid Method 488
12.6.1 Smoothing properties 490
12.6.2 The Coarse Grid Correction (CGC) method for linear problems 492
12.6.3 The two-grid iteration method for linear problems 496
12.6.4 The multi-grid method for linear problems 497
12.6.5 The multi-grid method for non-linear problems 499

APPENDIX: THOMAS ALGORITHM FOR TRIDIAGONAL SYSTEMS 505

A.1 Scalar Tridiagonal Systems 505
A.2 Periodic Tridiagonal Systems 506

INDEX 509