

Contents

<i>Preface</i>	xv
1. Some preliminaries: the standard discrete system	1
1.1 Introduction	1
1.2 The structural element and the structural system	4
1.3 Assembly and analysis of a structure	8
1.4 The boundary conditions	9
1.5 Electrical and fluid networks	10
1.6 The general pattern	12
1.7 The standard discrete system	14
1.8 Transformation of coordinates	15
References	16
2. A direct approach to problems in elasticity	18
2.1 Introduction	18
2.2 Direct formulation of finite element characteristics	19
2.3 Generalization to the whole region	26
2.4 Displacement approach as a minimization of total potential energy	29
2.5 Convergence criteria	31
2.6 Discretization error and convergence rate	32
2.7 Displacement functions with discontinuity between elements	33
2.8 Bound on strain energy in a displacement formulation	34
2.9 Direct minimization	35
2.10 An example	35
2.11 Concluding remarks	37
References	37
3. Generalization of the finite element concepts. Galerkin-weighted residual and variational approaches	39
3.1 Introduction	39
3.2 Integral or 'weak' statements equivalent to the differential equations	42
3.3 Approximation to integral formulations	46
3.4 Virtual work as the 'weak form' of equilibrium equations for analysis of solids or fluids	53

3.5	Partial discretization	55
3.6	Convergence	58
3.7	What are 'variational principles'?	60
3.8	'Natural' variational principles and their relation to governing differential equations	62
3.9	Establishment of natural variational principles for linear, self-adjoint differential equations	66
3.10	Maximum, minimum, or a saddle point?	69
3.11	Constrained variational principles. Lagrange multipliers and adjoint functions	70
3.12	Constrained variational principles. Penalty functions and the least square method	76
3.13	Concluding remarks	82
	References	84
4.	Plane stress and plane strain	87
4.1	Introduction	87
4.2	Element characteristics	87
4.3	Examples – an assessment of performance	97
4.4	Some practical applications	100
4.5	Special treatment of plane strain with an incompressible material	110
4.6	Concluding remark	111
	References	111
5.	Axisymmetric stress analysis	112
5.1	Introduction	112
5.2	Element characteristics	112
5.3	Some illustrative examples	121
5.4	Early practical applications	123
5.5	Non-symmetrical loading	124
5.6	Axisymmetry – plane strain and plane stress	124
	References	126
6.	Three-dimensional stress analysis	127
6.1	Introduction	127
6.2	Tetrahedral element characteristics	128
6.3	Composite elements with eight nodes	134
6.4	Examples and concluding remarks	135
	References	139
7.	Steady-state field problems – heat conduction, electric and magnetic potential, fluid flow, etc.	140
7.1	Introduction	140
7.2	The general quasi-harmonic equation	141
7.3	Finite element discretization	143
7.4	Some economic specializations	144
7.5	Examples – an assessment of accuracy	146
7.6	Some practical applications	149

7.7	Concluding remarks	161
	References	161
8.	'Standard' and 'hierarchical' element shape functions: some general families of C_0 continuity	164
8.1	Introduction	164
8.2	Standard and hierarchical concepts	165
8.3	Rectangular elements – some preliminary considerations	168
8.4	Completeness of polynomials	171
8.5	Rectangular elements – Lagrange family	172
8.6	Rectangular elements – 'serendipity' family	174
8.7	Elimination of internal variables before assembly – substructures	177
8.8	Triangular element family	179
8.9	Line elements	183
8.10	Rectangular prisms – Lagrange family	184
8.11	Rectangular prisms – 'serendipity' family	185
8.12	Tetrahedral elements	186
8.13	Other simple three-dimensional elements	190
8.14	Hierarchic polynomials in one dimension	190
8.15	Two- and three-dimensional, hierarchic, elements of the 'rectangle' or 'brick' type	193
8.16	Triangle and tetrahedron family	193
8.17	Global and local finite element approximation	196
8.18	Improvement of conditioning with hierarchic forms	197
8.19	Concluding remarks	198
	References	198
9.	Mapped elements and numerical integration – 'infinite' and 'singularity' elements	200
9.1	Introduction	200
9.2	Use of 'shape functions' in the establishment of coordinate transformations	203
9.3	Geometrical conformability of elements	206
9.4	Variation of the unknown function within distorted, curvilinear elements. Continuity requirements	206
9.5	Evaluation of element matrices (transformation in ξ, η, ζ coordinates)	208
9.6	Element matrices. Area and volume coordinates	211
9.7	Convergence of elements in curvilinear coordinates	213
9.8	Numerical integration – one-dimensional	217
9.9	Numerical integration – rectangular (2D) or right prism (3D) regions	219
9.10	Numerical integration – triangular or tetrahedral regions	221
9.11	Required order of numerical integration	223
9.12	Generation of finite element meshes by mapping. Blending functions	226
9.13	Infinite domains and infinite elements	229
9.14	Singular elements by mapping for fracture mechanics, etc.	234

9.15	A computational advantage of numerically integrated finite elements	236
9.16	Some practical examples of two-dimensional stress analysis	237
9.17	Three-dimensional stress analysis	238
9.18	Symmetry and repeatability	244
	References	246
10.	The patch test, reduced integration, and non-conforming elements	250
10.1	Introduction	250
10.2	Convergence requirements	251
10.3	The simple patch test (tests A and B) – a necessary condition for convergence	253
10.4	Generalized patch test (test C) and the single-element test	255
10.5	The generality of a numerical patch test	257
10.6	Higher order patch tests	257
10.7	Application of the patch test to plane elasticity elements with ‘standard’ and ‘reduced’ quadrature	258
10.8	Application of the patch test to an incompatible element	264
10.9	Generation of incompatible shape functions which satisfy the patch test	268
10.10	The weak patch test – example	270
10.11	Higher order patch test – assessment of robustness	271
10.12	Conclusion	273
	References	274
11.	Mixed formulation and constraints– complete field methods	276
11.1	Introduction	276
11.2	Discretization of mixed forms – some general remarks	278
11.3	Stability of mixed approximation. The patch test	280
11.4	Two-field mixed formulation in elasticity	284
11.5	Three-field mixed formulations in elasticity	291
11.6	An iterative method solution of mixed approximations	298
11.7	Complementary forms with direct constraint	301
11.8	Concluding remarks – mixed formulation or a test of element ‘robustness’	304
	References	304
12.	Incompressible materials, mixed methods and other procedures of solution	307
12.1	Introduction	307
12.2	Deviatoric stress and strain, pressure and volume change	307
12.3	Two-field incompressible elasticity ($u-p$ form)	308
12.4	Three-field nearly incompressible elasticity ($u-p-\varepsilon_v$ form)	314
12.5	Reduced and selective integration and its equivalence to penalized mixed problems	318
12.6	A simple iterative solution process for mixed problems: Uzawa method	323

12.7	Stabilized methods for some mixed elements failing the incompressibility patch test	326
12.8	Concluding remarks	342
	References	343
13.	Mixed formulation and constraints – incomplete (hybrid) field methods, boundary/Trefftz methods	346
13.1	General	346
13.2	Interface traction link of two (or more) irreducible form subdomains	346
13.3	Interface traction link of two or more mixed form subdomains	349
13.4	Interface displacement ‘frame’	350
13.5	Linking of boundary (or Trefftz)-type solution by the ‘frame’ of specified displacements	355
13.6	Subdomains with ‘standard’ elements and global functions	360
13.7	Lagrange variables or discontinuous Galerkin methods?	361
13.8	Concluding remarks	361
	References	362
14.	Errors, recovery processes and error estimates	365
14.1	Definition of errors	365
14.2	Superconvergence and optimal sampling points	370
14.3	Recovery of gradients and stresses	375
14.4	Superconvergent patch recovery – SPR	377
14.5	Recovery by equilibration of patches – REP	383
14.6	Error estimates by recovery	385
14.7	Other error estimators – residual based methods	387
14.8	Asymptotic behaviour and robustness of error estimators – the Babuška patch test	392
14.9	Which errors should concern us?	398
	References	398
15.	Adaptive finite element refinement	401
15.1	Introduction	401
15.2	Some examples of adaptive h -refinement	404
15.3	p -refinement and hp -refinement	415
15.4	Concluding remarks	426
	References	426
16.	Point-based approximations; element-free Galerkin – and other meshless methods	429
16.1	Introduction	429
16.2	Function approximation	431
16.3	Moving least square approximations – restoration of continuity of approximation	438
16.4	Hierarchical enhancement of moving least square expansions	443
16.5	Point collocation – finite point methods	446

16.6	Galerkin weighting and finite volume methods	451
16.7	Use of hierarchic and special functions based on standard finite elements satisfying the partition of unity requirement	457
16.8	Closure	464
	References	464
17.	The time dimension – semi-discretization of field and dynamic problems and analytical solution procedures	468
17.1	Introduction	468
17.2	Direct formulation of time-dependent problems with spatial finite element subdivision	468
17.3	General classification	476
17.4	Free response – eigenvalues for second-order problems and dynamic vibration	477
17.5	Free response – eigenvalues for first-order problems and heat conduction, etc.	484
17.6	Free response – damped dynamic eigenvalues	484
17.7	Forced periodic response	485
17.8	Transient response by analytical procedures	486
17.9	Symmetry and repeatability	490
	References	491
18.	The time dimension – discrete approximation in time	493
18.1	Introduction	493
18.2	Simple time-step algorithms for the first-order equation	495
18.3	General single-step algorithms for first- and second-order equations	508
18.4	Multistep recurrence algorithms	522
18.5	Some remarks on general performance of numerical algorithms	530
18.6	Time discontinuous Galerkin approximation	536
18.7	Concluding remarks	538
	References	538
19.	Coupled systems	542
19.1	Coupled problems – definition and classification	542
19.2	Fluid–structure interaction (Class I problem)	545
19.3	Soil–pore fluid interaction (Class II problems)	558
19.4	Partitioned single-phase systems – implicit–explicit partitions (Class I problems)	565
19.5	Staggered solution processes	567
	References	572
20.	Computer procedures for finite element analysis	576
20.1	Introduction	576
20.2	Data input module	578
20.3	Memory management for array storage	588
20.4	Solution module – the command programming language	590
20.5	Computation of finite element solution modules	597

20.6	Solution of simultaneous linear algebraic equations	609
20.7	Extension and modification of computer program <i>FEAPpv</i>	618
	References	618
Appendix A:	Matrix algebra	620
Appendix B:	Tensor-indicial notation in the approximation of elasticity problems	626
Appendix C:	Basic equations of displacement analysis	635
Appendix D:	Some integration formulae for a triangle	636
Appendix E:	Some integration formulae for a tetrahedron	637
Appendix F:	Some vector algebra	638
Appendix G:	Integration by parts in two and three dimensions (Green's theorem)	643
Appendix H:	Solutions exact at nodes	645
Appendix I:	Matrix diagonalization or lumping	648
	Author index	655
	Subject index	663