Contents

Preface xiii

Acknowledgments xvii

Part I Symmetry in Convex Systems

1. MONO-DUALITY IN STATIC SYSTEMS 3

1.1 The First Problem in The Calculus of Variations 4

1.1.1 Elastic String: Tri-Canonical Forms and Total Potentials 4

1.1.2 Variational Problems and Kinematically Admissible Spaces 8

1.2 Fundamental Lemma and Euler Equation 12

1.2.1 Convex Functionals and Gâteaux Variations 12

1.2.2 Primal Variational Problems and Euler Equations 17

1.3 Linear Operators and Bilinear Forms 20

1.3.1 Lagrange Identity and Weighted Bilinear Forms 21

1.3.2 Boundary Operators and Conditions 23

1.4 Legendre Transformation and Duality 27

1.4.1 Legendre Duality in Euclidean Geometry 27

1.4.2 Legendre Conjugate Transformation 30

1.5 Alternative Variational Problems: Lagrange Equations and Multipliers 33

1.5.1 Boundary-Value Problems and Minimum Potential Energy Principles 34

1.5.2 Complementary Energy Principles and Statically Admissible Spaces 35

1.6 Saddle Lagrange Duality Theory 38

1.6.1 Saddle-Lagrangian and Minimax Theorems 38

1.6.2 Weak and Strong Duality Theorems 42

1.7 Applications and Commentary 48

1.7.1 Applications in Convex Optimization and Boundary-Value Problems 48

1.7.2 Historical Notes and Commentary 56

2. BI-DUALITY IN DYNAMICAL SYSTEMS 59

2.1 Particle Dynamics: Newton and Einstein 60

2.1.1 Newton's Triality Law 60

2.1.2 Einstein's Relativity 61

2.2 Convex Hamiltonian Systems 63

2.2.1 Total Action and Hamilton Principle 63

2.2.2 Hamiltonian and Canonical Forms 64
2.3 Least Action Principle: Legendre and Jacobian Condition 65
2.3.1 Euler Equation and Legendre's Condition 65
2.3.2 Jacobi Condition and Least Action Principle 66
2.4 Initial-Value Problems and Dissipative Hamiltonian Systems 69
2.4.1 Bilinear Forms in Initial-Value Problems 69
2.4.2 Framework for Linear and Nonlinear Dissipative Systems 70
2.5 Complementary Hamiltonian Principles and Euler-Lagrange Equations 74
2.5.1 Stabilities and Extremum Action Principles 75
2.5.2 Lagrangian and Complementary Actions 78
2.6 Super-Lagrangian Duality 81
2.6.1 Super Lagrangian and Super-Critical Points 82
2.6.2 Bi-Duality Theory 85
2.7 Applications and Commentary 89
2.7.1 Nonconvex Optimization and Variational Problem 89
2.7.2 Commentary 94

Part II Symmetry Breaking: Triality Theory in Nonconvex Systems

3. TRI-DUALITY IN NONCONVEX SYSTEMS 99
3.1 Constitutive Symmetry Breaking in Convex Systems 100
3.1.1 Legendre Duality Breaking: KKT conditions 100
3.1.2 Duality Restoration: Finite Deformation Measures 103
3.1.3 Nonsmooth Constitutive Laws and Mirror Materials 106
3.2 Geometrical Symmetry Breaking: Framework in Nonconvex Systems 110
3.2.1 Static Bifurcation: van de Waals Double-Well Energy 110
3.2.2 Post-buckling: von Kármán's Paradox and Extended Beam Theory 113
3.2.3 Dynamical Bifurcation: Duffing Equation and Its Canonical Form 117
3.3 Quadratic Canonical Transformation and the Gap Functional 121
3.3.1 Nonconvex Boundary-Value Problems and Duality Gap 121
3.3.2 Canonical Dual Transformation and Operator Decomposition 123
3.3.3 Canonical Boundary-Value Problem and Complementary Gap Functional 125
3.4 Complementary Energy Variational Principle and Analytic Solutions 127
3.4.1 Dual Problem and Algebraic Euler-Lagrange Equation 127
3.4.2 Analytic Solution Theorem 130
3.4.3 Quadratic Canonical Energy and Cubic Algebraic Curve 132
3.5 Tri-Extremum Principles and Triality Theory 136
3.5.1 Nonlinear Lagrangian and Critical Point Theory 137
3.5.2 Triality Theorems 140
3.5.3 Tri-Duality Theory 141
3.6 Canonical Dual Transformation for Nonconvex Dynamical Systems 146
3.6.1 Spacetime geometrical measure and Canonical Dynamical Equations 146
3.6.2 Triality Theory in Nonconvex Hamilton Systems 148
3.6.3 Duality for Duffing System 151
3.7 Applications and Commentary 157
3.7.1 Dissipative Systems and Nonsmooth Bifurcation Problems 157

3.7.2 Historical Notes and Commentary 163

4. MULTI-DUALITY AND CLASSIFICATIONS OF GENERAL SYSTEMS 167
4.1 The First Type of Sequential Canonical Dual Transformation 168
4.1.1 Nonconvex Lagrangian and Discrete Dual Problems 168
4.1.2 Second-Order Lagrangian and Coupled Dual Problem 171
4.1.3 Sequential Nonlinear Lagrangian Dual Transformations 172
4.2 The Second Type of Sequential Canonical Dual Transformation 173
4.2.1 General Nonconvexity and Composite Transformation 174
4.2.2 Canonical Lagrangian and Analytic Solution 175
4.2.3 Multi-Well Energy and Algebraic Curves 176
4.3 Canonical Systems: The Classifications 179
4.3.1 Elementary System and Fundamental Equation 179
4.3.2 Canonical Systems and Classifications 180
4.3.3 Polar Systems 186
4.4 Generalized Tri-Duality Principles 191
4.5 Framework for Geometrically Linear Canonical Systems 194
4.5.1 Discrete Systems: Electrical Networks and Structural Mechanics 194
4.5.2 Continuum Mechanics and Stress Functions 199
4.5.3 Coupled Electromagnetic Field: Maxwell's equations 204
4.6 Framework for Boundary-Value Problems 206
4.6.1 Sobolev Spaces 206
4.6.2 Boundary Value Problems 207
4.6.3 Formal Adjoints and Bilinear Concomitants 209
4.7 Nonlinear Systems and Commentary 212
4.7.1 Superconductivity: Ginzburg-Landau Equation 212
4.7.2 Commentary and Bibliographic Notes 214

Part III Duality in Canonical Systems

5. DUALITY IN GEOMETRICALLY LINEAR SYSTEMS 219
5.1 Extended Variational Problems and Fenchel Duality Theory 220
5.1.1 Extended Functionals and Semicontinuities 220
5.1.2 Primal Variational Problems and General Theorems 228
5.1.3 Fenchel Transformation 232
5.1.4 Generalized Differentials and Fenchel Duality 236
5.1.5 Calculus on Banach Spaces 240
5.2 Perturbation and Rockafellar Duality Theory 243
5.2.1 Perturbation, Normality and Stability 243
5.2.2 Rockafellar Transformation and Duality Theory 246
5.3 Extended Lagrange Duality Theory 252
5.3.1 Lagrangian Forms and Critical Points 253
5.3.2 Extended Saddle Lagrangian Duality Theory 256
5.3.3 Super- and Sub-Lagrangian Duality 258
5.3.4 Duality in Finite-dimensional Systems and Index Theory 259
5.4 Hamilton and Clarke Duality Theories 261
5.4.1 Hamiltonian and Canonical Inclusions 261
5.4.2 Clarke Duality Theorem 266

5.5 Duality in Variational Inequality and Complementarity Problems 268
5.5.1 Primal Variational Inequality Problems 268
5.5.2 Complementarity and Karush-Kuhn-Tucker Conditions 271
5.5.3 Bi-Complementarity and Duality 274
5.5.4 Mathematical Programming and Primal-Dual Methods 276
5.5.5 Commentary 281

6. DUALITY IN FINITE DEFORMATION SYSTEMS 283
6.1 Finite Deformation Theory 284
6.1.1 Deformation Geometry and Equilibrium Principles 284
6.1.2 Hyperelasticity and Constitutive Laws 287
6.1.3 Boundary-Value Problems, Regularity and Uniqueness 292
6.2 Primal, Dual and Polar Variational Problems 295
6.2.1 Generalized Convexities 295
6.2.2 Fenchel-Rockafellar Dual Variational Principle and Duality Gap 299
6.2.3 Polar Variational Problems 301
6.3 Canonical Strain Measures and Complementary Gap Functional 303
6.3.1 Canonical Strain Measures and Canonical Boundary-Value Problems 304
6.3.2 Hill-Seth Strain Family and Hill's Conjugate Pairing 307
6.3.3 Generalized Variational Principles in Continuum Mechanics 311
6.4 Tri-Duality Theory in Finite Deformation Problems 314
6.4.1 Triality Theorem 314
6.4.2 Tri-Duality Theorems in Finite Deformation Theory 317
6.4.3 Pure Complementary Energies and Analytical Solutions 320
6.5 Minimal Hyper-Surface Problems 324
6.5.1 Parametric Minimal Hypersurface and Primal-Dual Problems 324
6.5.2 Polar-Complementary Variational Problem and Polar Surfaces 328
6.5.3 Non-Parametric Surfaces 335
6.6 Applications and Commentary 339
6.6.1 3-D Elastic Cylindrical Tube 339
6.6.2 Deformation Theory with Internal Variables 341
6.6.3 Historical Remarks and Commentary 343

7. APPLICATIONS, OPEN PROBLEMS AND CONCLUDING REMARKS 347
7.1 Constitutive Nonlinearity: Plastic Limit Analysis 348
7.1.1 Constitutive Law and Super-Potentials 348
7.1.2 Complementary Bounding Theorems 350
7.1.3 Penalty-Duality Methods and Lower Bound Theorems 351
7.2 Contact Problems of Extended Elastoplastic Beam Theory 356
7.2.1 Extended Beam Model and Contact Problem 356
7.2.2 Elastoplastic Contact Problem and Bi-Complementarity 358
7.2.3 Dual Variational Inequality and Applications 361
7.3 Geometrical Nonlinearity: von Kármán Plate 365
7.3.1 Large Deformation Thin Plate and Primal Problem 365
7.3.2 Canonical Dual Transformation 368
7.3.3 Complementary Variational Principles 370
7.4 Large Deformation Beam Theory 372
7.4.1 Extended Beam Model and the Reason for von Kármán's Paradox 372
7.4.2 Extended Second-Order Dynamic Beam Model 375

7.4.3 Complementary Energy Variational Principles 378
7.5 Optimal Shape Designs and Eigenvalue Problems 381
7.5.1 Optimal Shape Design of Beam Theory 381
7.5.2 Eigenvalue Problems on Extremum Surfaces 384
7.5.3 Singularities for Coupled Eigenvalue Problems on Extremum Surfaces 386
7.6 Miscellaneous Open Problems 390
7.6.1 Potential Korteweg-deVries Equation 390
7.6.2 Potential Boussinesq Equation 391
7.6.3 Invariant Nonlinear Problems 392
7.7 Commentary and Concluding Remarks 394
7.7.1 The Seventh Commentary 394
7.7.2 Concluding Remarks 398

Appendices 401
A–Duality in Linear Analysis 401
A.1 Linear spaces and duality 401
A.2 Bilinear Forms and Inner Product Spaces 408
A.3 Linear functionals and Dual spaces 410
B–Linear Operators and Adjointness 416
B.1 Linear Operators 416
B.2 Adjoint Operators 418
B.3 Duality Relations for Range and Nullspace 421
C–Nonlinear Operators 422
C.1 Operators on Finite-Dimensional Spaces 424
C.2 Monotone and Pseudo-Monotone Operators on Banach Spaces 426
C.3 Potential Operators and Duality Mappings 427

References 433

Index 449