

## Contents

Preface *xiv*

About the Companion Website *xviii*

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Physical Processes and Mathematical Models	1
1.2	Approximation, Error, and Convergence	3
1.3	Approximate Solution of Differential Equations and the Finite Element Method	5
1.4	Brief History of the Finite Element Method	6
1.5	Finite Element Software	8
1.6	Significance of Finite Element Analysis for Engineering	8
1.7	Typical Process for Obtaining a Finite Element Solution for a Physical Problem	12
1.8	A Note on Linearity and the Principle of Superposition	14
	References	16
<b>2</b>	<b>Strong and Weak Form for One-Dimensional Problems</b>	<b>17</b>
2.1	Strong Form for One-Dimensional Elasticity Problems	17
2.2	General Expressions for Essential and Natural B.C. in One-Dimensional Elasticity Problems	23
2.3	Weak Form for One-Dimensional Elasticity Problems	24
2.4	Equivalence of Weak Form and Strong Form	28
2.5	Strong Form for One-Dimensional Heat Conduction	32
2.6	Weak Form for One-Dimensional Heat Conduction	37
	Problems	44
	References	46
<b>3</b>	<b>Finite Element Formulation for One-Dimensional Problems</b>	<b>47</b>
3.1	Introduction—Piecewise Approximation	47
3.2	Shape (Interpolation) Functions and Finite Elements	51
3.3	Discrete Equations for Piecewise Finite Element Approximation	59
3.4	Finite Element Equations for Heat Conduction	66
3.5	Accounting for Nodes with Prescribed Solution Value (“Fixed” Nodes)	67
3.6	Examples on One-Dimensional Finite Element Analysis	68
3.7	Numerical Integration—Gauss Quadrature	91
3.8	Convergence of One-Dimensional Finite Element Method	100

3.9	Effect of Concentrated Forces in One-Dimensional Finite Element Analysis	106
	Problems	108
	References	111
<b>4</b>	<b>Multidimensional Problems: Mathematical Preliminaries</b>	<b>112</b>
4.1	Introduction	112
4.2	Basic Definitions	113
4.3	Green's Theorem—Divergence Theorem and Green's Formula	118
4.4	Procedure for Multidimensional Problems	121
	Problems	122
	References	122
<b>5</b>	<b>Two-Dimensional Heat Conduction and Other Scalar Field Problems</b>	<b>123</b>
5.1	Strong Form for Two-Dimensional Heat Conduction	123
5.2	Weak Form for Two-Dimensional Heat Conduction	129
5.3	Equivalence of Strong Form and Weak Form	131
5.4	Other Scalar Field Problems	133
5.4.1	Two-Dimensional Potential Fluid Flow	133
5.4.2	Fluid Flow in Porous Media	137
5.4.3	Chemical (Molecular) Diffusion-Reaction	138
	Problems	139
<b>6</b>	<b>Finite Element Formulation for Two-Dimensional Scalar Field Problems</b>	<b>141</b>
6.1	Finite Element Discretization and Piecewise Approximation	141
6.2	Three-Node Triangular Finite Element	148
6.3	Four-Node Rectangular Element	153
6.4	Isoparametric Finite Elements and the Four-Node Quadrilateral (4Q) Element	158
6.5	Numerical Integration for Isoparametric Quadrilateral Elements	165
6.6	Higher-Order Isoparametric Quadrilateral Elements	176
6.7	Isoparametric Triangular Elements	178
6.8	Continuity and Completeness of Isoparametric Elements	181
6.9	Concluding Remarks: Finite Element Analysis for Other Scalar Field Problems	183
	Problems	184
	References	188
<b>7</b>	<b>Multidimensional Elasticity</b>	<b>189</b>
7.1	Introduction	189
7.2	Definition of Strain Tensor	189
7.3	Definition of Stress Tensor	191
7.4	Representing Stress and Strain as Column Vectors—The Voigt Notation	193
7.5	Constitutive Law (Stress-Strain Relation) for Multidimensional Linear Elasticity	194
7.6	Coordinate Transformation Rules for Stress, Strain, and Material Stiffness Matrix	199

7.7	Stress, Strain, and Constitutive Models for Two-Dimensional (Planar) Elasticity	202
7.8	Strong Form for Two-Dimensional Elasticity	208
7.9	Weak Form for Two-Dimensional Elasticity	212
7.10	Equivalence between the Strong Form and the Weak Form	215
7.11	Strong Form for Three-Dimensional Elasticity	218
7.12	Using Polar (Cylindrical) Coordinates	220
	References	225
<b>8</b>	<b>Finite Element Formulation for Two-Dimensional Elasticity</b>	<b>226</b>
8.1	Piecewise Finite Element Approximation—Assembly Equations	226
8.2	Accounting for Restrained (Fixed) Displacements	231
8.3	Postprocessing	232
8.4	Continuity—Completeness Requirements	232
8.5	Finite Elements for Two-Dimensional Elasticity	232
8.5.1	Three-Node Triangular Element (Constant Strain Triangle)	233
8.5.2	Quadrilateral Isoparametric Element	237
8.5.3	Example: Calculation of Stiffness Matrix and Equivalent Nodal Forces for Four-Node Quadrilateral Isoparametric Element	245
	Problems	251
<b>9</b>	<b>Finite Element Formulation for Three-Dimensional Elasticity</b>	<b>257</b>
9.1	Weak Form for Three-Dimensional Elasticity	257
9.2	Piecewise Finite Element Approximation—Assembly Equations	258
9.3	Isoparametric Finite Elements for Three-Dimensional Elasticity	264
9.3.1	Eight-Node Hexahedral Element	264
9.3.2	Numerical (Gaussian) Quadrature for Hexahedral Isoparametric Elements	272
9.3.3	Calculation of Boundary Integral Contributions to Nodal Forces	276
9.3.4	Higher-Order Hexahedral Isoparametric Elements	277
9.3.5	Tetrahedral Isoparametric Elements	277
9.3.6	Three-Dimensional Elements from Collapsed (Degenerated) Hexahedral Elements	280
9.3.7	Concluding Remark: Continuity and Completeness Ensured by Three-Dimensional Isoparametric Elements and Use for Other Problems	281
	Problems	287
	Reference	288
<b>10</b>	<b>Topics in Applied Finite Element Analysis</b>	<b>289</b>
10.1	Concentrated Loads in Multidimensional Analysis	289
10.2	Effect of Autogenous (Self-Induced) Strains—The Special Case of Thermal Strains	291
10.3	The Patch Test for Verification of Finite Element Analysis Software	294
10.4	Subparametric and Superparametric Elements	295
10.5	Field-Dependent Natural Boundary Conditions: Emission Conditions and Compliant Supports	296
10.6	Treatment of Nodal Constraints	302



10.7	Treatment of Compliant (Spring) Connections Between Nodal Points	309
10.8	Symmetry in Analysis	311
10.9	Axisymmetric Problems and Finite Element Analysis	316
10.10	A Brief Discussion on Efficient Mesh Refinement Problems	319
	References	323
<b>11</b>	<b>Convergence of Multidimensional Finite Element Analysis, Locking Phenomena in Multidimensional Solids and Reduced Integration</b>	<b>324</b>
11.1	Convergence of Multidimensional Finite Elements	324
11.2	Effect of Element Shape in Multidimensional Analysis	327
11.3	Incompatible Modes for Quadrilateral Finite Elements	328
11.4	Volumetric Locking in Continuum Elements	332
11.5	Uniform Reduced Integration and Spurious Zero-Energy (Hourglass) Modes	337
11.6	Resolving the Problem of Hourglass Modes: Hourglass Stiffness	339
11.7	Selective-Reduced Integration	346
11.8	The B-bar Method for Resolving Locking Problems	348
	References	352
<b>12</b>	<b>Multifield (Mixed) Finite Elements</b>	<b>353</b>
12.1	Multifield Weak Forms for Elasticity	354
12.2	Mixed (Multifield) Finite Element Formulations	359
12.3	Two-Field (Stress-Displacement) Formulations and the Pian-Sumihara Quadrilateral Element	367
12.4	Displacement-Pressure ( $u$ - $p$ ) Formulations and Finite Element Approximations	370
12.5	Stability of Mixed $u$ - $p$ Formulations—the inf-sup Condition	374
12.6	Assumed (Enhanced)-Strain Methods and the B-bar Method as a Special Case	377
12.7	A Concluding Remark for Multifield Elements	381
	References	382
<b>13</b>	<b>Finite Element Analysis of Beams</b>	<b>383</b>
13.1	Basic Definitions for Beams	383
13.2	Differential Equations and Boundary Conditions for 2D Beams	385
13.3	Euler-Bernoulli Beam Theory	388
13.4	Strong Form for Two-Dimensional Euler-Bernoulli Beams	392
13.5	Weak Form for Two-Dimensional Euler-Bernoulli Beams	394
13.6	Finite Element Formulation: Two-Node Euler-Bernoulli Beam Element	397
13.7	Coordinate Transformation Rules for Two-Dimensional Beam Elements	404
13.8	Timoshenko Beam Theory	408
13.9	Strong Form for Two-Dimensional Timoshenko Beam Theory	411
13.10	Weak Form for Two-Dimensional Timoshenko Beam Theory	411
13.11	Two-Node Timoshenko Beam Finite Element	415
13.12	Continuum-Based Beam Elements	417
13.13	Extension of Continuum-Based Beam Elements to General Curved Beams	424

13.14	Shear Locking and Selective-Reduced Integration for Thin Timoshenko Beam Elements	440
	Problems	443
	References	446
<b>14</b>	<b>Finite Element Analysis of Shells</b>	<b>447</b>
14.1	Introduction	447
14.2	Stress Resultants for Shells	451
14.3	Differential Equations of Equilibrium and Boundary Conditions for Flat Shells	452
14.4	Constitutive Law for Linear Elasticity in Terms of Stress Resultants and Generalized Strains	456
14.5	Weak Form of Shell Equations	464
14.6	Finite Element Formulation for Shell Structures	472
14.7	Four-Node Planar (Flat) Shell Finite Element	480
14.8	Coordinate Transformations for Shell Elements	485
14.9	A “Clever” Way to Approximately Satisfy $C^1$ Continuity Requirements for Thin Shells—The Discrete Kirchhoff Formulation	500
14.10	Continuum-Based Formulation for Nonplanar (Curved) Shells	510
	Problems	521
	References	522
<b>15</b>	<b>Finite Elements for Elastodynamics, Structural Dynamics, and Time-Dependent Scalar Field Problems</b>	<b>523</b>
15.1	Introduction	523
15.2	Strong Form for One-Dimensional Elastodynamics	525
15.3	Strong Form in the Presence of Material Damping	527
15.4	Weak Form for One-Dimensional Elastodynamics	529
15.5	Finite Element Approximation and Semi-Discrete Equations of Motion	530
15.6	Three-Dimensional Elastodynamics	536
15.7	Semi-Discrete Equations of Motion for Three-Dimensional Elastodynamics	539
15.8	Structural Dynamics Problems	539
15.8.1	Dynamic Beam Problems	540
15.8.2	Dynamic Shell Problems	543
15.9	Diagonal (Lumped) Mass Matrices and Mass Lumping Techniques	546
15.9.1	Mass Lumping for Continuum (Solid) Elements	546
15.9.2	Mass Lumping for Structural Elements (Beams and Shells)	548
15.10	Strong and Weak Form for Time-Dependent Scalar Field (Parabolic) Problems	549
15.10.1	Time-Dependent Heat Conduction	549
15.10.2	Time-Dependent Fluid Flow in Porous Media	552
15.10.3	Time-Dependent Chemical Diffusion	554
15.11	Semi-Discrete Finite Element Equations for Scalar Field (Parabolic) Problems	555
15.12	Solid and Structural Dynamics as a “Parabolic” Problem: The State-Space Formulation	557
	Problems	558
	References	559

<b>16</b>	<b>Analysis of Time-Dependent Scalar Field (Parabolic) Problems</b>	<b>560</b>
16.1	Introduction	560
16.2	Single-Step Algorithms	562
16.3	Linear Multistep Algorithms	568
16.3.1	Adams-Bashforth (AB) Methods	569
16.3.2	Adams-Moulton (AM) Methods	569
16.4	Predictor-Corrector Algorithms—Runge-Kutta (RK) Methods	569
16.5	Convergence of a Time-Stepping Algorithm	572
16.5.1	Stability of Time-Stepping Algorithms	572
16.5.2	Error, Order of Accuracy, Consistency, and Convergence	574
16.6	Modal Analysis and Its Use for Determining the Stability for Systems with Many Degrees of Freedom	583
	Problems	587
	References	587
<b>17</b>	<b>Solution Procedures for Elastodynamics and Structural Dynamics</b>	<b>588</b>
17.1	Introduction	588
17.2	Modal Analysis: What Will NOT Be Presented in Detail	589
17.2.1	Proportional Damping Matrices—Rayleigh Damping Matrix	592
17.3	Step-by-Step Algorithms for Direct Integration of Equations of Motion	594
17.3.1	Explicit Central Difference Method	595
17.3.2	Newmark Method	597
17.3.3	Hilber-Hughes Taylor (HHT or Alpha) Method	599
17.3.4	Stability and Accuracy of Transient Solution Algorithms	601
17.4	Application of Step-by-Step Algorithms for Discrete Systems with More than One Degrees of Freedom	608
	Problems	613
	References	613
<b>18</b>	<b>Verification and Validation for the Finite Element Method</b>	<b>615</b>
18.1	Introduction	615
18.2	Code Verification	615
18.2.1	Order of Accuracy Testing	616
18.2.2	Systematic Mesh Refinement	617
18.2.3	Exact Solutions	618
18.3	Solution Verification	622
18.3.1	Iterative Error	623
18.3.2	Discretization Error	624
18.4	Numerical Uncertainty	627
18.5	Sources and Types of Uncertainty	629
18.6	Validation Experiments	630
18.7	Validation Metrics	631
18.8	Extrapolation of Model Prediction Uncertainty	633
18.9	Predictive Capability	634
	References	634



<b>19</b>	<b>Numerical Solution of Linear Systems of Equations</b>	<b>637</b>
19.1	Introduction	637
19.2	Direct Methods	638
19.2.1	Gaussian Elimination	638
19.2.2	The LU Decomposition	639
19.3	Iterative Methods	640
19.3.1	The Jacobi Method	642
19.3.2	The Conjugate Gradient Method	642
19.4	Parallel Computing and the Finite Element Method	644
19.4.1	Efficiency of Parallel Algorithms	645
19.4.2	Parallel Architectures	647
19.5	Parallel Conjugate Gradient Method	649
	References	653

## **Appendix A: Concise Review of Vector and Matrix Algebra 654**

A.1	Preliminary Definitions	654
A.1.1	Matrix Example	655
A.1.2	Vector Equality	655
A.2	Matrix Mathematical Operations	656
A.2.1	Exterior Product	657
A.2.2	Product of Two Matrices	657
A.2.3	Inverse of a Square Matrix	660
A.2.4	Orthogonal Matrix	660
A.3	Eigenvalues and Eigenvectors of a Matrix	660
A.4	Rank of a Matrix	662

## **Appendix B: Review of Matrix Analysis for Discrete Systems 664**

B.1	Truss Elements	664
B.2	One-Dimensional Truss Analysis	666
B.3	Solving the Global Stiffness Equations of a Discrete System and Postprocessing	671
B.4	The ID Array Concept (for Equation Assembly)	673
B.5	Fully Automated Assembly: The Connectivity (LM) Array Concept	680
B.6	Advanced Interlude—Programming of Assembly When the Restrained Degrees of Freedom Have Nonzero Values	682
B.7	Advanced Interlude 2: Algorithms for Postprocessing	683
B.8	Two-Dimensional Truss Analysis—Coordinate Transformation Equations	684
B.9	Extension to Three-Dimensional Truss Analysis	693
	Problems	694

## **Appendix C: Minimum Potential Energy for Elasticity—Variational Principles 695**

## **Appendix D: Calculation of Displacement and Force Transformations for Rigid-Body Connections 700**