

Contents

Notation.....	11
Introduction.....	17
Chapter 1. Homotopy depth.....	25
1. Relative homotopy groups.....	26
1.1. Relative q -cells and relative homotopy groups.....	26
1.2. The case of “0-cells”.....	28
1.3. Boundary operator and induced homomorphism.....	28
1.4. Higher connectedness.....	29
2. The Seifert–van Kampen theorem.....	29
2.1. Presentation of groups.....	29
2.2. Amalgamated free product.....	30
2.3. The Seifert–van Kampen theorem.....	31
3. Prill neighbourhoods.....	32
4. Blakers–Massey neighbourhoods.....	34
5. Homotopy groups and stratified transversality.....	36
5.1. Whitney stratifications.....	36
5.2. Thom–Mather first isotopy theorem and conic structure theorem.....	38
5.3. Connectedness theorems and stratified transversality.....	38
5.4. Proof of Theorem 5.6.....	39
6. Rectified homotopy depth.....	43
6.1. Homotopy depth of polyhedra.....	43
6.2. Rectified homotopy depth of complex analytic spaces.....	46
Chapter 2. Lefschetz–Zariski hyperplane section theorems.....	49
1. The classical hyperplane section theorems of Lefschetz and Zariski.....	51
2. Hyperplane section theorems for singular quasi-projective varieties.....	53
2.1. The case of non-singular varieties.....	54
2.2. The case of singular varieties.....	55
3. The Lefschetz–Zariski–van Kampen pencil method.....	57
4. Proof of the singular hyperplane section theorem.....	59
5. Proof of the singular version of the theorem on pencils of hyperplane sections	60
5.1. Reduction to the case $\Sigma \subseteq Z$	60

5.2. Blowing up along the pencil base locus and fibration outside the special hyperplanes.....	61
5.3. Homotopy of the pair of bundles $(\tilde{X}^*, \tilde{A}^*)$	62
5.4. Moving the relative cells of $F^q(\tilde{X}, \tilde{X}^*, \gamma_0)$ to general position.....	63
5.5. Deforming the relative cells of $F^q(\tilde{X}, \tilde{X}^*, \gamma_0)$ into cells of $F^q(\tilde{X}^* \cup \tilde{A}, \tilde{X}^*, \gamma_0)$	65
5.6. Comparing the homotopy groups of the pairs $(\tilde{X}, \tilde{X}^*)$ and $(\tilde{A}, \tilde{A}^*)$	67
5.7. Coming back to the original space X	69
5.8. End of proof of Theorem 3.1.....	71
Chapter 3. Zariski–van Kampen theorems.....	74
1. Monodromy variation operators.....	75
1.1. Setting.....	75
1.2. Monodromy.....	77
1.3. Standard monodromy variation operator.....	79
1.4. Relative monodromy variation operator.....	79
2. The classical Zariski–van Kampen theorem.....	80
3. Zariski–van Kampen theorems for singular varieties.....	81
3.1. The first singular version.....	81
3.2. The special case of non-singular varieties.....	82
3.3. The approach via relative monodromy variation.....	83
4. Proofs of the singular Zariski–van Kampen theorems.....	85
4.1. Reduction to the case $\Sigma \subseteq \mathbb{Z}$	85
4.2. Blowing up along the pencil base locus, fibration outside the special hyperplanes and monodromy variation on the blow up space.....	86
4.3. The fundamental group $\pi_1(\tilde{X}^*, (x_0, \lambda))$	88
4.4. The fundamental group $\pi_1(\tilde{X}, (x_0, \lambda))$	89
4.5. End of proofs of Theorems 3.1 and 3.5.....	98
Chapter 4. Fundamental groups of plane curve complements.....	99
1. Quick review of the Zariski–van Kampen pencil method.....	100
1.1. The fundamental group $\pi_1(\mathbb{P}^2 \setminus C)$	100
1.2. The fundamental group $\pi_1(\mathbb{C}^2 \setminus C)$	102
1.3. Relation between $\pi_1(\mathbb{P}^2 \setminus C)$ and $\pi_1(\mathbb{C}^2 \setminus C)$	103
2. Fundamental group and first homology.....	104
3. Degeneration principle.....	105
4. Zariski’s classical results on non-singular curves.....	106
5. Nodal curves and the Zariski–Fulton–Deligne theorem.....	110
6. On certain cuspidal and nodal-cuspidal curves: Nori’s theorem.....	112
7. Curves of low degree.....	113
8. Oka–Sakamoto product formula.....	118
9. Non-atypical curves.....	121
9.1. Bifurcation set, critical values and atypical values.....	121

9.2. Non-atypical curves and their fundamental groups.....	122
9.3. Examples.....	124
10. Zariski pairs.....	127
Chapter 5. Fundamental groups of complements to plane $\mathbb{R}$ -join-type curves.....	132
1. Singular points of join-type curves.....	133
2. The special groups $G(p; q)$ and $G(p; q; r)$	134
2.1. The group $G(p; q)$	134
2.2. The group $G(p; q; r)$	136
2.3. Criteria of commutativity for $G(p; q)$ and $G(p; q; r)$	138
3. Generic and semi-generic $\mathbb{R}$ -join-type curves.....	139
4. $\mathbb{R}$ -join-type curves which are not semi-generic.....	143
5. Bifurcation graphs.....	146
6. Calculation of the fundamental groups.....	151
6.1. Generic curves (proof of Theorem 3.3).....	151
6.2. Semi-generic curves (proof of Theorem 3.6).....	159
6.3. The first class of non-semi-generic curves (proof of Theorem 4.1).....	162
6.4. The second class of non-semi-generic curves (proof of Theorem 4.4)...	164
Chapter 6. Zariski–van Kampen theorems for higher homotopy groups of hypersurface complements.....	167
1. Homotopy groups of complements to hypersurfaces with isolated singularities via homology.....	169
1.1. Setting.....	169
1.2. Universal covering of $X := \mathbb{P}^n \setminus H$	171
1.3. Homotopy of $X := \mathbb{P}^n \setminus H$ via the homology of the universal covering $X' := H' \setminus j(H)$	172
2. Zariski–van Kampen theorem for higher homotopy groups: approach via “degeneration”.....	174
2.1. Homology degeneration operators on the total space X'_{λ_i} of the covering $\rho: X'_{\lambda_i} \rightarrow X_{\lambda_i}$	175
2.2. Homotopy degeneration operator on X_{λ_i}	177
2.3. Zariski–van Kampen theorem via degeneration.....	179
3. Zariski–van Kampen theorem for higher homotopy groups: approach via relative monodromy variation.....	179
3.1. Relative monodromy variation operators via homology.....	180
3.2. Zariski–van Kampen theorem via relative variation (first version).....	182
3.3. Homotopy definition of the relative monodromy variation operators..	183
3.4. Zariski–van Kampen theorem via relative variation (second version)...	189
3.5. Conjectures.....	189
3.6. Zariski–van Kampen theorem for higher homology groups of non-singular quasi-projective varieties.....	190
4. The second Lefschetz theorem.....	191
4.1. The classical second Lefschetz theorem.....	191

4.2. Picard–Lefschetz formula and hard Lefschetz theorem.....	193
4.3. A counter-example for quasi-projective varieties.....	194
4.4. A modified statement for non-singular quasi-projective varieties.....	195
5. Proof of the Zariski–van Kampen theorem for higher homology groups....	200
References.....	206
Index.....	215