

# Contents

## 1 Description of the Deformation and Kinematics of a Saturated Porous Medium. Mass Conservation 1

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