

Contents

Preface

xiii

1	The standard discrete system and origins of the finite element method	1
1.1	Introduction	1
1.2	The structural element and the structural system	3
1.3	Assembly and analysis of a structure	5
1.4	The boundary conditions	6
1.5	Electrical and fluid networks	7
1.6	The general pattern	9
1.7	The standard discrete system	10
1.8	Transformation of coordinates	11
1.9	Problems	13
2	A direct physical approach to problems in elasticity: plane stress	19
2.1	Introduction	19
2.2	Direct formulation of finite element characteristics	20
2.3	Generalization to the whole region – internal nodal force concept abandoned	31
2.4	Displacement approach as a minimization of total potential energy	34
2.5	Convergence criteria	37
2.6	Discretization error and convergence rate	38
2.7	Displacement functions with discontinuity between elements – non-conforming elements and the patch test	39
2.8	Finite element solution process	40
2.9	Numerical examples	40
2.10	Concluding remarks	46
2.11	Problems	47
3	Generalization of the finite element concepts. Galerkin-weighted residual and variational approaches	54
3.1	Introduction	54
3.2	Integral or ‘weak’ statements equivalent to the differential equations	57
3.3	Approximation to integral formulations: the weighted residual-Galerkin method	60

3.4	Virtual work as the 'weak form' of equilibrium equations for analysis of solids or fluids	69
3.5	Partial discretization	71
3.6	Convergence	74
3.7	What are 'variational principles'?	76
3.8	'Natural' variational principles and their relation to governing differential equations	78
3.9	Establishment of natural variational principles for linear, self-adjoint, differential equations	81
3.10	Maximum, minimum, or a saddle point?	83
3.11	Constrained variational principles. Lagrange multipliers	84
3.12	Constrained variational principles. Penalty function and perturbed lagrangian methods	88
3.13	Least squares approximations	92
3.14	Concluding remarks – finite difference and boundary methods	95
3.15	Problems	97
4	'Standard' and 'hierarchical' element shape functions: some general families of C_0 continuity	103
4.1	Introduction	103
4.2	Standard and hierarchical concepts	104
4.3	Rectangular elements – some preliminary considerations	107
4.4	Completeness of polynomials	109
4.5	Rectangular elements – Lagrange family	110
4.6	Rectangular elements – 'serendipity' family	112
4.7	Triangular element family	116
4.8	Line elements	119
4.9	Rectangular prisms – Lagrange family	120
4.10	Rectangular prisms – 'serendipity' family	121
4.11	Tetrahedral elements	122
4.12	Other simple three-dimensional elements	125
4.13	Hierarchic polynomials in one dimension	125
4.14	Two- and three-dimensional, hierarchical elements of the 'rectangle' or 'brick' type	128
4.15	Triangle and tetrahedron family	128
4.16	Improvement of conditioning with hierarchical forms	130
4.17	Global and local finite element approximation	131
4.18	Elimination of internal parameters before assembly – substructures	132
4.19	Concluding remarks	134
4.20	Problems	134
5	Mapped elements and numerical integration – 'infinite' and 'singularity elements'	138
5.1	Introduction	138
5.2	Use of 'shape functions' in the establishment of coordinate transformations	139
5.3	Geometrical conformity of elements	143
5.4	Variation of the unknown function within distorted, curvilinear elements. Continuity requirements	143

5.5	Evaluation of element matrices. Transformation in ξ, η, ζ coordinates	145
5.6	Evaluation of element matrices. Transformation in area and volume coordinates	148
5.7	Order of convergence for mapped elements	151
5.8	Shape functions by degeneration	153
5.9	Numerical integration – one dimensional	160
5.10	Numerical integration – rectangular (2D) or brick regions (3D)	162
5.11	Numerical integration – triangular or tetrahedral regions	164
5.12	Required order of numerical integration	164
5.13	Generation of finite element meshes by mapping. Blending functions	169
5.14	Infinite domains and infinite elements	170
5.15	Singular elements by mapping – use in fracture mechanics, etc.	176
5.16	Computational advantage of numerically integrated finite elements	177
5.17	Problems	178
6	Problems in linear elasticity	187
6.1	Introduction	187
6.2	Governing equations	188
6.3	Finite element approximation	201
6.4	Reporting of results: displacements, strains and stresses	207
6.5	Numerical examples	209
6.6	Problems	217
7	Field problems – heat conduction, electric and magnetic potential and fluid flow	229
7.1	Introduction	229
7.2	General quasi-harmonic equation	230
7.3	Finite element solution process	233
7.4	Partial discretization – transient problems	237
7.5	Numerical examples – an assessment of accuracy	239
7.6	Concluding remarks	253
7.7	Problems	253
8	Automatic mesh generation	264
8.1	Introduction	264
8.2	Two-dimensional mesh generation – advancing front method	266
8.3	Surface mesh generation	286
8.4	Three-dimensional mesh generation – Delaunay triangulation	303
8.5	Concluding remarks	323
8.6	Problems	323
9	The patch test, reduced integration, and non-conforming elements	329
9.1	Introduction	329
9.2	Convergence requirements	330
9.3	The simple patch test (tests A and B) – a necessary condition for convergence	332
9.4	Generalized patch test (test C) and the single-element test	334
9.5	The generality of a numerical patch test	336
9.6	Higher order patch tests	336

9.7	Application of the patch test to plane elasticity elements with 'standard' and 'reduced' quadrature	337
9.8	Application of the patch test to an incompatible element	343
9.9	Higher order patch test – assessment of robustness	347
9.10	Concluding remarks	347
9.11	Problems	350
10	Mixed formulation and constraints – complete field methods	356
10.1	Introduction	356
10.2	Discretization of mixed forms – some general remarks	358
10.3	Stability of mixed approximation. The patch test	360
10.4	Two-field mixed formulation in elasticity	363
10.5	Three-field mixed formulations in elasticity	370
10.6	Complementary forms with direct constraint	375
10.7	Concluding remarks – mixed formulation or a test of element 'robustness'	379
10.8	Problems	379
11	Incompressible problems, mixed methods and other procedures of solution	383
11.1	Introduction	383
11.2	Deviatoric stress and strain, pressure and volume change	383
11.3	Two-field incompressible elasticity (\mathbf{u} - p form)	384
11.4	Three-field nearly incompressible elasticity (\mathbf{u} - p - ϵ_v form)	393
11.5	Reduced and selective integration and its equivalence to penalized mixed problems	398
11.6	A simple iterative solution process for mixed problems: Uzawa method	404
11.7	Stabilized methods for some mixed elements failing the incompressibility patch test	407
11.8	Concluding remarks	421
11.9	Problems	422
12	Multidomain mixed approximations – domain decomposition and 'frame' methods	429
12.1	Introduction	429
12.2	Linking of two or more subdomains by Lagrange multipliers	430
12.3	Linking of two or more subdomains by perturbed lagrangian and penalty methods	436
12.4	Interface displacement 'frame'	442
12.5	Linking of boundary (or Trefftz)-type solution by the 'frame' of specified displacements	445
12.6	Subdomains with 'standard' elements and global functions	451
12.7	Concluding remarks	451
12.8	Problems	451
13	Errors, recovery processes and error estimates	456
13.1	Definition of errors	456
13.2	Superconvergence and optimal sampling points	459
13.3	Recovery of gradients and stresses	465

13.4	Superconvergent patch recovery – SPR	467
13.5	Recovery by equilibration of patches – REP	474
13.6	Error estimates by recovery	476
13.7	Residual-based methods	478
13.8	Asymptotic behaviour and robustness of error estimators – the Babuška patch test	488
13.9	Bounds on quantities of interest	490
13.10	Which errors should concern us?	494
13.11	Problems	495
14	Adaptive finite element refinement	500
14.1	Introduction	500
14.2	Adaptive h -refinement	503
14.3	p -refinement and hp -refinement	514
14.4	Concluding remarks	518
14.5	Problems	520
15	Point-based and partition of unity approximations. Extended finite element methods	525
15.1	Introduction	525
15.2	Function approximation	527
15.3	Moving least squares approximations – restoration of continuity of approximation	533
15.4	Hierarchical enhancement of moving least squares expansions	538
15.5	Point collocation – finite point methods	540
15.6	Galerkin weighting and finite volume methods	546
15.7	Use of hierarchic and special functions based on standard finite elements satisfying the partition of unity requirement	549
15.8	Concluding remarks	558
15.9	Problems	558
16	The time dimension – semi-discretization of field and dynamic problems and analytical solution procedures	563
16.1	Introduction	563
16.2	Direct formulation of time-dependent problems with spatial finite element subdivision	563
16.3	General classification	570
16.4	Free response – eigenvalues for second-order problems and dynamic vibration	571
16.5	Free response – eigenvalues for first-order problems and heat conduction, etc.	576
16.6	Free response – damped dynamic eigenvalues	578
16.7	Forced periodic response	579
16.8	Transient response by analytical procedures	579
16.9	Symmetry and repeatability	583
16.10	Problems	584
17	The time dimension – discrete approximation in time	589
17.1	Introduction	589

17.2	Simple time-step algorithms for the first-order equation	590
17.3	General single-step algorithms for first- and second-order equations	600
17.4	Stability of general algorithms	609
17.5	Multistep recurrence algorithms	615
17.6	Some remarks on general performance of numerical algorithms	618
17.7	Time discontinuous Galerkin approximation	619
17.8	Concluding remarks	624
17.9	Problems	626
18	Coupled systems	631
18.1	Coupled problems – definition and classification	631
18.2	Fluid–structure interaction (Class I problems)	634
18.3	Soil–pore fluid interaction (Class II problems)	645
18.4	Partitioned single-phase systems – implicit–explicit partitions (Class I problems)	653
18.5	Staggered solution processes	655
18.6	Concluding remarks	660
19	Computer procedures for finite element analysis	664
19.1	Introduction	664
19.2	Pre-processing module: mesh creation	664
19.3	Solution module	666
19.4	Post-processor module	666
19.5	User modules	667
	Appendix A: Matrix algebra	668
	Appendix B: Tensor-indicial notation in the approximation of elasticity problems	674
	Appendix C: Solution of simultaneous linear algebraic equations	683
	Appendix D: Some integration formulae for a triangle	692
	Appendix E: Some integration formulae for a tetrahedron	693
	Appendix F: Some vector algebra	694
	Appendix G: Integration by parts in two or three dimensions (Green’s theorem)	699
	Appendix H: Solutions exact at nodes	701
	Appendix I: Matrix diagonalization or lumping	704
	Author index	711
	Subject index	719